Tilings of Low-Genus Surfaces by Quadrilaterals

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Abstract

In contribution to the classification of all tilings of low-genus surfaces, the kaleidoscopic and non-kaleidoscopic tilings by quadrilaterals are given up to genus 12. As part of their classification, the algebraic structure of the conformal tiling groups and the geometric structure of the tiles are specified. In addition, several infinite classes of tilings and tiling groups are presented.

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1 Introduction

1.1 Tilings on surfaces

A *surface* S is a two-dimensional object that locally looks like the plane. Every surface may be represented topologically by a sphere with handles attached. The number of handles of a given surface is called the *genus* of the surface and is denoted σ. A *tiling* T of a surface S is a non-overlapping covering of the surface by polygons or tiles. In this paper, we refer exclusively to tilings by quadrilaterals. As an example, consider Figure 1. This surface is a torus with σ=1 and is tiled by rectangles. For a tiling on a genus 1 surface, the tiles are Euclidean polygons, and thus the tiling can be constructed in the Euclidean plane. However, when $\sigma > 1$, the tiles are non-Euclidean.

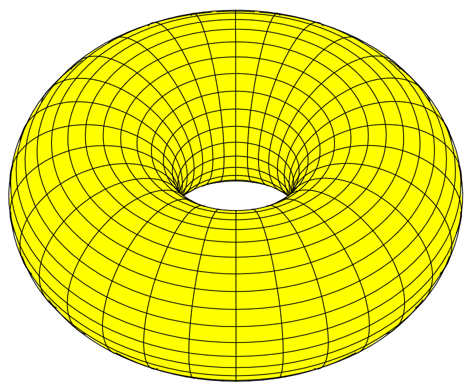

Figure 1: (2,2,2,2) tiling of the torus.

1.2 Geometric considerations

For a surface with $\sigma \geq 2$, a hyperbolic tiling can be constructed in the unit complex disk and then mapped to the surface by the universal covering map. The details of the mapping are not important. The key is that the geometry of the tiles and tiling is hyperbolic. In hyperbolic geometry, the concepts of angle sums, lines, distances, etc. are non-Euclidean. For example, a quadrilateral tile would be constructed by four segments of circles perpendicular to the unit disk, and the sum of the four angles would be less than 2π. For further background on hyperbolic geometry and tilings see [1] and [4].

A tiling is called *kaleidoscopic* if the mirror reflection in the edge of any tile extends to a global isometry of the surface mapping tiles to tiles. The geometric structure of the tiles, such as angle measure and distance, is preserved under such a reflection. Specifications for a kaleidoscopic tiling will be further discussed in Section 2.2.

A tiling is called *geodesic* if each edge is part of a smooth, closed curve which is a union of edges. If the tiling is kaleidoscopic then the curve must be a geodesic on the surface. If a tiling is geodesic, and kaleidoscopic then it follows that there are an even number of tiles meeting in congruent angles at each vertex, though the number of tiles may depend on the vertex. So if there are $2k$ angles meeting at a vertex for some integer k, then each angle has measure $\frac{2\pi}{2k} = \frac{\pi}{k}$. In order to simplify notation, a quadrilateral with angles measuring $\frac{\pi}{k}$, $\frac{\pi}{l}$, $\frac{\pi}{m}$, and $\frac{\pi}{n}$ is referred to as a (k, l, m, n) quadrilateral. Mathematically, the most interesting tilings are those with the highest degree of symmetry. Thus, in this paper only geodesic, kaleidoscopic tilings are considered.

We may extend these definitions to geodesic, kaleidoscopic tilings of the hyperbolic plane.

2 Tilings and tiling groups

For background on this section see [4].

Often the geometry of a tiling is described by algebraic structures allowing for more concrete calculations. The geometric properties and actions of a tiling can be represented using group theory.

2.1 Full and conformal tiling groups

The reflections in the edges of a tiling generate a group of isometries of the tiling, called the tiling group. We describe here the tiling group of a tiling by quadrilaterals. Generalizing to tilings by polygons can be done given this discussion. For any kaleidoscopically tiled plane or surface, it is easy to show that every tile in the plane or surface is the image, by some element of the tiling group, of a single tile, called the master tile. In Figure 2, the master tile is pictured, and labelled Ω_0. The sides of the master tile are labelled p, q, r, and s, and we also denote by p, q, r, and s the individual reflections in the corresponding sides. We also see that Ω_0 is a (k, l, m, n)-quadrilateral. The various reflected tiles $p\Omega_0, \ldots, s\Omega_0$ are also pictured. At each of the vertices of the quadrilateral, the product of the two reflections in the sides of the quadrilateral meeting at the vertex is a rotation fixing the vertex. The angle of rotation is twice the angle at this vertex. For example the product $a = pq$ is a reflection first through q then through p, which is equivalent to a counter-clockwise rotation through $\frac{2\pi}{k}$ radians as in Figure 2. Rotations around each of the other corners can be defined in the same way, so that $b = qr$, $c = rs$, and $d = sp$ are counterclockwise rotations through $\frac{2\pi}{l}$, $\frac{2\pi}{m}$, and $\frac{2\pi}{n}$ radians, respectively.

From the geometry of the master tile, we can derive relations among these group elements. The reflections have the property

$$o(p) = o(q) = o(r) = o(s) = 2.$$

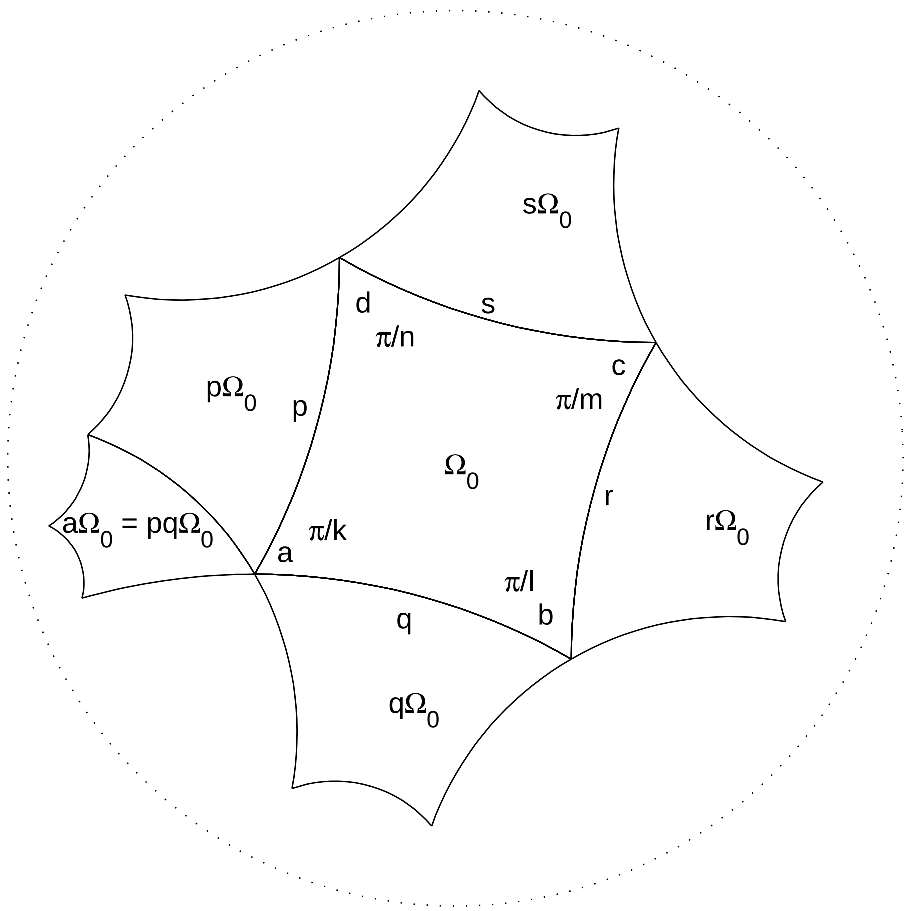

Figure 2: The master tile and generators of G and G^*

The rotations have the properties

$$o(a) = k, o(b) = l, o(c) = m, o(d) = n, \tag{1}$$

and

$$abcd = pqqrrssp = 1. \tag{2}$$

Let $G^* = \langle p, q, r, s \rangle$ and $G = \langle a, b, c, d \rangle = \langle a, b, c \rangle$ be the groups generated by the above elements. The group G^* is called the (full) tiling group (or the reflection group determined Ω_0 since it is generated by reflections in the sides of Ω_0). The group G is the subgroup of index 2 of orientation-preserving elements of G^* and is called the *conformal tiling group* (or the rotation group determined Ω_0 since it is generated by rotations at the vertices of Ω_0). For elements a, b, c, and d with properties (1) and (2) that generate the conformal tiling group, the quadruple (a, b, c, d) is called a *generating* (k, l, m, n)-*quadruple.*

Remark 1 *It is possible to have a locally kaleidoscopic tiling in which any two tiles are mirror images of each other in their common edge, but the local isometries do not extend globally. In this case the local rotations at vertices will also exist. It is also possible that each of these local rotations may extend to the entire surface, so that there is a conformal tiling group but no full tiling group.*

Remark 2 *If the tiling is on the hyperbolic plane we will denote the conformal and full tiling groups by* Λ *and* Λ^*, *respectively.*

In the first major work on automorphisms of surfaces by Hurwitz [6], we find the following formula, known as the Riemann-Hurwitz equation, relating the genus of the surface, the order of the conformal tiling group, and the orders of the generating rotations (it can be proved using the Euler characteristic).

$$\frac{2\sigma - 2}{|G|} = 2 - (\frac{1}{k} + \frac{1}{l} + \frac{1}{m} + \frac{1}{n}) \tag{3}$$

Theorem 1 (Hurwitz [6]) *Let* G *be a group. Suppose that* (a, b, c, d) *is a* (k, l, m, n)-*generating quadruple for* G *and that* σ *obtained from (3) is a nonnegative integer, and that* Ω_0 *is any* (k, l, m, n)-*quadrilateral. Then* G *is (isomorphic to) a conformal tiling group of a geodesic, locally kaleidoscopic tiling of a surface* S *of genus* σ *by* (k, l, m, n)-*quadrilaterals, congruent to* Ω_0, *though the tiling need not be kaleidoscopic. The tiling may be constructed so that the quadruple* (a, b, c, d) *are the rotations at the vertices of some appropriately chosen quadrilateral, with appropriate ordering. Furthermore, if* $(a', b', c', d') = \omega \cdot (a, b, c, d)$ *for some* $\omega \in Aut(G)$ *then the surface* S' *constructed from* (a', b', c', d') *and* Ω_0 *is conformally equivalent to* S *except that the* G-*action on* S' *is twisted by* ω.

2.2 Automorphism groups and the kaleidoscopic condition

The mirror reflection over a geodesic as described in the definition of kaleidoscopic is algebraically represented by conjugation of the elements of G by the reflection q. Such conjugation action induces an automorphism, θ, on G, such that

$$\begin{aligned} \theta(a) &= qaq^{-1} = qpqq = qp = a^{-1} \\ \theta(b) &= qbq^{-1} = qqrq = rq = b^{-1} \\ \theta(c) &= qcq^{-1} = qrsq = qrsrrq = bc^{-1}b^{-1} \\ \theta(d) &= qcq^{-1} = qspq = qrrspssrrq = bcd^{-1}c^{-1}b^{-1} \\ \theta^2 &= 1. \end{aligned} \tag{4}$$

The following theorem extends from triangles to quadrilaterals, a group theoretic characterization of when a tiling is kaleidoscopic (see [9]).

Theorem 2 *If a group G and (k,l,m,n)-quadrilateral Ω_0) satisfy the hypothesis of Theorem 1 and there exists an automorphism $\theta \in Aut(G)$ that satisfies*

$$\begin{aligned} \theta(a) &= a^{-1} \\ \theta(b) &= b^{-1} \\ \theta(c) &= bc^{-1}b^{-1} \\ \theta(d) &= bcd^{-1}c^{-1}b^{-1} \end{aligned} \tag{5}$$

then there is a surface with a kaleidoscopic, geodesic tiling by (k,l,m,n)-quadrilaterals, congruent to Ω_0, a conformal tiling group, G, and a full tiling group, $G^* \simeq \langle\theta\rangle \ltimes G$, such that

$$p = \theta a^{-1}, q = \theta, r = \theta b, s = \theta bc. \tag{6}$$

The elements p,q,r,s, and a,b,c,d are the reflections in the sides and the rotations at the vertices, respectively, of an appropriately chosen tile, with appropriate ordering.

Remark 3 *The group $\langle\theta\rangle \ltimes G$ formally consists of elements of the form $\{(\theta^i, g), i = 0, 1, g \in G\}$, with multiplication*

$$(\theta^i, g_1) \cdot (\theta^j, g_2) = (\theta^{i+j}, \theta^j(g_1)g_2)$$

Informally the elements of $\langle\theta\rangle \ltimes G$ are of the form g and θg, $g \in G$ with the non-obvious multiplications being $g\theta h = \theta g^\theta h$ and $\theta g\theta h = g^\theta h$, where $g^\theta = \theta(g)$.

3 Organization of the classification process

Before outlining the method used to find geodesic and geodesic, kaleidoscopic tilings of low genus surfaces, some known results concerning possible values of (k,l,m,n) and $|G|$ for a given σ should be presented.

Theorem 3 (Harvey [5]) *Let G be a conformal tiling group with a generating (k,l,m,n) quadruple, acting on a surface of genus σ. Then:*

$$|G| \leq 12(\sigma - 1) \tag{7}$$

$$k,l,m,n \mid |G| \tag{8}$$

$$k,l,m,n \leq 4\sigma + 2. \tag{9}$$

Equation (7) follows from equation (3) by observing that the right hand side achieves its smallest positive value when three of $\{k,l,m,n\}$ equal 2 and the other equals 3. Equation

(9) was originally proven by proven by Wiman, a more recent proof may be found in [5].

3.1 Classification outline

Now suppose the genus σ is fixed. Here are the steps of the algorithm.
Step 1. Construct a list of feasible $|G|$ and (k, l, m, n).
Step 2. For each entry in the list found in Step 1, search through the MAGMA/GAP database for a group with a (k, l, m, n) generating quadruple. If such a group G exists, we have a geodesic tiling.
Step 3. Given a group G, find the involutions in $\mathrm{Aut}(G)$.
Step 4. Determine if any of the representatives found in Step 3 satisfy the conditions 5.

To begin classification for specific genus, we find all possible values for $|G|$ and (k, l, m, n) by using the Riemann-Hurwitz equation. The corresponding vectors are called *branching data* and have been enumerated in a Maple script bradatpoly.mws available at the website [13]. So we start with this list of data, and we must eliminate those for which no group exists and find all groups that satisfy each branching data. The conditions (7) - (9), which limit the possible $|G|$ and (k, l, m, n), are easily implemented in the Maple computer search for branching data.

3.2 Representing groups and automorphism groups

The search for conformal tiling groups of a given branching data (Step 2 of previous section) is done in MAGMA. The computer algebra systems MAGMA [12] and GAP [11] have several ways of representing finite groups depending on different group properties. For our purposes, using MAGMA's power conjugate representation of solvable groups (see Section 4) is sufficient, except for some rare cases for which MAGMA's permutation group representation for non-solvable groups is used. Thus, G is always represented in this fashion except when being used in calculations involving Aut(G). In this case, elements of G are represented in the same way as automorphisms of G, in their Cayley representation.
The motivation in using Cayley representation comes from the following known propositions:

Proposition 4 *(see e.g., [3]) Let $G \subseteq \Sigma$ be a pair of groups. Let $N' = N_\Sigma(G)$ and $Z = Z_\Sigma(G)$ be the normalizer and centralizer of G in Σ respectively. Let $Ad : N' \to Ad(G)$ be the adjoint homomorphism defined by $Ad_a(g)=aga^{-1}$. Then Ad maps N'/Z isomorphically onto a subgroup M of Aut(G) containing the inner automorphisms of G,*

$$Inn(G) \subseteq M \subseteq Aut(G).$$

Furthermore, if Σ is the Cayley representation of G in the symmetric group on $|G|$ elements, $M = Aut(G)$ (see [8]).

Proposition 5 *(see e.g., [3]) Let Σ_G be the group of permutations on the elements of G. For $g \in G$, let L_g be the element of Σ_G defined by $L_g(x) = gx \; \forall \; x \in G$. Let $G_L = \{L_g : g \in G\}$ and $N = N_{\Sigma_G}(G_L)$ be the normalizer of G_L in the symmetric group on $|G|$ elements. Then,*

$$N \simeq Aut(G) \ltimes G = Hol(G), (\textit{the holomorph of } G).$$

To find the Cayley representation of G, the set $\{L_g$: g a generator of G$\}$ is first calculated with L_g as defined above, using a minimal or near minimal number of generators. This set contains the generators of the Cayley representation in $S_{|G|}$, the symmetric group on $|G|$ elements. Thus, Cay_G, the subgroup of $S_{|G|}$ generated by this set, is the Cayley representation of G. If we calculate $N = N_{S_{|G|}}(Cay_G)$, the normalizer of the Cayley representation of G in $S_{|G|}$, we get the result of Proposition 5. From Proposition 4 and the usage of the Cayley representation of G as Σ, we see that Aut(G) is isomorphic to a quotient of this N. Thus, any desired automorphism of G has an homomorphic image in N, and we represent Aut(G) by N, using it as described in Section 3.3.

3.3 Finding generating quadruples

In performing Step 2 from Section 3.1, suppose there is a given branching data $[|G|, (k, l, m, n)]$ and candidate G. The possible generating quadruples are

$$\begin{aligned} X_G(k, l, m, n) \;\; = \;\; & \{(a, b, c, d) \in G^4 : o(a) = k, o(b) = l, o(c) = m, o(d) = n, \\ & abcd = 1, G = \langle a, b, c, d \rangle\}. \end{aligned}$$

Thus, if this set is non-empty for a candidate G, then G is a conformal tiling group for the according branching data. This condition can be tested with usually much less calculation time by first constructing the sets

$$O(s) = \{g \in G : o(g) = s\},$$
$$O_R(s) = \{g \in Rep(G) : o(g) = s\}$$

where Rep(G) is the set of representatives of the G conjugacy classes, for $s = k, l, m$, and n, and then constructing

$$X'_G(k, l, m, n) = \{(a, b, c, d) \in O_R(k) \times O(l) \times O(m) \times O(n) : abcd = 1, G = \langle a, b, c, d \rangle\}.$$

This set is generally much smaller than the full set of generating quadruples, but constructing and testing this set is sufficient because $X_G(k, l, m, n) = \{\} \iff X'_G(k, l, m, n) = \{\}$.

3.4 Finding an appropriate automorphism of G

In the search for an appropriate automorphism of G, as mentioned in Step 4 of Section 3.1, representatives of the conjugacy classes of N are tested, for this is quicker due to not having to calculate completely $\mathrm{Aut}(G)$. In using N in lieu of $\mathrm{Aut}(G)$, the conditions (4) need to be carefully applied. The condition that $\theta^2 = 1$ limits θ to being the identity automorphism or an automorphism of order 2. Let $\phi : N \to N/Z(G) = \mathrm{Aut}(G)$ be the homomorphism taking N into $\mathrm{Aut}(G)$. Then an easily constructed set that includes all elements of N satisfying this condition can be found by noting the following. Let

$$\Theta = \{\theta \in \mathrm{Aut}(G) : \theta^2 = 1\} = \{\phi(n) : n \in N, \phi(n)^2 = 1\},$$

then,

$$\{\phi(n) : n, n = 1 \ \ or \ \ o(n) = 2\} \subseteq \Theta \subseteq \{\phi(n) : n \in N, n = 1 \ \ or \ \ 0 \equiv o(n) \ mod \ 2\}.$$

This implies that using $N^* = \{n : n \in N : n = 1 \ \ or \ \ 0 \equiv o(n) \ mod \ 2\}$ will suffice.

Finding sufficient equivalent tests for the remaining conditions in (5) can be done by using a proposition in [4], which gives conditions for the conjugation action of an element of N to be the desired automorphism of G. If we let the subscript c denote the Cayley representation of an element, then conjugation by $n \in N^*$ is an automorphism of G satisfying Theorem 2 if and only if

$$\begin{aligned} n a_c n^{-1} &= a_c^{-1}, \\ n b_c n^{-1} &= b_c^{-1}, \quad \text{and} \\ n c_c n^{-1} &= b_c c_c^{-1} b_c^{-1}. \end{aligned}$$

Testing for the existence of such an n leads to the conclusion of whether or not the tiling in question is kaleidoscopic.

4 Categorizing group structure

The MAGMA database contains 174,365 groups. This includes all groups of order $n \leq 1000$, except those of order 512 or 768. 99.88% of these groups are solvable and are stored in power-conjugate (PC) representation and the others are stored as permutation groups. It is therefore necessary to understand power-conjugate notation in order to see the structure of each conformal tiling group G. Following is a description of how to extract group structure from PC representation.

For any solvable group G, we have

$$G = \langle a_1, a_2, \ldots, a_n \rangle,$$

where $a_1, a_2, \ldots, a_n$ are generators of G with certain properties. Let $G_i = \langle a_i, \ldots, a_n \rangle$, then $G = G_1 \supset G_2 \supset \ldots \supset G_{n+1} = e$, and we may, WLOG, assume a proper inclusion at each step. As G is a solvable group, we may choose the generators to attain a stronger restriction, i.e. $G = G_1 \rhd G_2 \rhd \ldots \rhd G_{n+1} = \{e\}$. As a consequence, ${a_j}^{a_i} \in G$ for all $j > i$. In addition, if we assume that the (proper) chain above is as long as possible, then $|G_i/G_{i+1}| = p_i$, a prime, for $i = 1, \ldots, n$. It then follows that every element of G may be written in a unique form

$$g = {a_1}^{r_1}{a_2}^{r_2} \cdots {a_n}^{r_n},$$

where $0 \leq r_1 < p_1, ..., 0 \leq r_n < p_n$. In particular, the elements of G_i have the form ${a_i}^{r_i}{a_{i+1}}^{r_{i+1}} \cdots {a_n}^{r_n}$. Thus the relations among the elements of G all have the form

$$\begin{aligned} {a_i}^{p_i} &= w_i = w_i(a_{i+1}, ..., a_n) \in G \\ {a_j}^{a_i} &= w_{i,j} = w_{i,j}(a_{i+1}, ..., a_n) \in G_{i+1}, \end{aligned}$$

where w_i and $w_{i,j}$ are words that determine a PC presentation

$$G = \langle a_1, a_2, \ldots, a_n | {a_i}^{p_i} = w_i, {a_j}^{a_i} = w_{i,j} \rangle.$$

The integer n is determined by the number of prime factors of $|G|$, counted with multiplicity. Though $n(n+1)/2$ relations are required for the presentation of a group, in the MAGMA representation, the abelian relations are left out. Thus, for example, here are five non-isomorphic for groups of order 8, presented in MAGMA format. To be strictly correct we should include, for example, in the presentation of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ the relations $a_2^{a_1} = a_2$, $a_3^{a_1} = a_3$, and $a_3^{a_2} = a_3$.

$$\begin{aligned} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 &= \langle a_1, a_2, a_3 \mid {a_1}^2 = 1, {a_2}^2 = 1, {a_3}^2 = 1 \rangle \\ \mathbb{Z}_2 \times \mathbb{Z}_4 &= \langle a_1, a_2, a_3 \mid {a_1}^2 = a_3 \rangle \\ \mathbb{Z}_8 &= \langle a_1, a_2, a_3 \mid {a_1}^2 = a_2, {a_2}^2 = a_3 \rangle \\ \mathbb{D}_4 &= \langle a_1, a_2, a_3 \mid {a_1}^2 = a_3, {a_2}^{a_1} = a_2 a_3 \rangle \\ \mathbb{H} &= \langle a_1, a_2, a_3 \mid {a_1}^2 = a_3, {a_2}^2 = a_3, {a_2}^{a_1} = a_2 a_3 \rangle. \end{aligned}$$

In these examples, the distinction between abelian and non-abelian groups is clear. It may also be noted that all of these groups have the same number of generators, as $|G| = 8 = 2^3$. It is important to realize that, although $\mathbb{Z}_8$ *may* be generated by only one element, power-conjugate representation lists three generators.

5 Observations, examples, and propositions

While not taking the direct approach to the issue that we do here, several previous works give results leading to some existence theorems for conformal tiling groups. In this section, we will present some applications of these theorems, some classes of tilings, and some general properties of the classification of tilings found in Appendix A.

5.1 Cyclic and Abelian groups

Example 6 *Let's consider the branching data* $[4\sigma-2,(2,2,2,4\sigma-2)]$ *for a genus* σ *surface. We see that* $o(d)=|G|=4\sigma-2$*, and thus* G *must be cyclic. In [5], Harvey presents necessary and sufficient conditions for a cyclic group* G *to be a conformal tiling group. When applied to this example, these conditions show that there will not be a geodesic tiling with this branching data for any possible* σ*. This is interesting because this branching data passes the initial three conditions, (7)-(9), for the existence of a tiling, but yields no geodesic tiling, a rare occurrence.*

There is one large class of examples of conformal tilings groups which always result in kaleidoscopic tilings, namely abelian groups.

Proposition 7 *Let* (a,b,c,d) *be a generating* (k,l,m,n)*-quadruple of an abelian group* G*. Then any surface tiling generated by* (a,b,c,d) *is automatically kaleidoscopic.*

Proof By Proposition 5 we need only show that there is an automorphism θ of $\mathrm{Aut}(G)$ satisfying (5). However, for an Abelian group the map $\theta : g \to g^{-1}$ does the job. ■

5.2 Variation in the geometry

One class of tilings of particular interest consists of multiple tilings with the same $|G|$, and angles $\{\pi/k, \pi/l, \pi/m, \pi/n\}$. This can occur in two ways. First, two non-isomorphic groups of the same order could tile the same surface with the same quadrilateral tiles. In this case, the two resulting surfaces are known to be topologically equivalent, but conditions for conformal equivalence of the tilings are presently unknown. In most cases they are conformally inequivalent, e.g., if the G is the full automorphism group of S.

The other instance in which different tilings exist with equivalent branching data occurs when no three of the integers $k, l, m,$ and n corresponding to the angles of the quadrilateral tiles are equal. In this case, at least two combinatorially distinct non-symmetric quadrilaterals exist with angles $\frac{\pi}{k}, \frac{\pi}{l}, \frac{\pi}{m}$, and $\frac{\pi}{n}$. For example, as seen in Table 1 of section 6, two different tilings exist for the branching data $[4,(2,2,4,4)]$. Both tilings have conformal tiling group $\mathbb{Z}_4$, but one tiling is with quadrilaterals that have two adjacent $\frac{\pi}{2}$ angles while the other is with quadrilaterals that have equal opposite angles. To distinguish between the two tilings, we say one is with $(2,2,4,4)$ quadrilaterals while the other is with $(2,4,2,4)$ quadrilaterals.

Remark 4 *In a given tiling there are 8 ways in which the angle sequence of a tile can be traversed, four ways for clockwise-oriented tiles and 4 ways for their mirror images. This give 8 different generating quadruples for the action of the group G, per the following table.*

clockwise description	*generating quadruple*	*counterclockwise description*	*generating quadruple*
(k,l,m,n)	(a,b,c,d)	(k,n,m,l)	$(a,a^{-1}da,bcb^{-1},b)$
(l,m,n,k)	(b,c,d,a)	(l,k,n,m)	$(b,b^{-1}ab,cdc^{-1},c)$
(m,n,k,l)	(c,d,a,b)	(m,l,k,n)	$(c,c^{-1}bc,dad^{-1},d)$
(n,k,l,m)	(d,a,b,c)	(n,m,l,k)	$(d,d^{-1}cd,aba^{-1},a)$

For instance, third and fourth and columns of the first row are obtained by flipping the quadrilateral over the (k,l) side and moving around the reflected quadrilateral in a clockwise fashion starting at the k vertex. None of the quadruples in the second and fourth columns yield new surfaces or G-actions.

There are 24 ways of writing out the angle data broken up into three sets of 8. The three sets of 8 are determined by what vertex lies diagonally opposite the k-vertex. Indeed, the three sets of 8 are the orbit under the dihedral group of a regular 4-gon considered as a subgroup Σ_4. The three sets of 8 can yield geometrically different quadrilaterals as noted above. For each of the 24 orderings of the angle data we can associate a generating quadruple. Using the above table we need only do it for a representative of each of the sets of 8. Here they are in table form.

geometric type	*generating quadruple*
(k,l,m,n)	(a,b,c,d)
(k,m,l,n)	$(a,c,c^{-1}bc,d)$
(k,l,n,m)	$(a,b,d,d^{-1}cd)$

Proposition 8 *For a given branching data $[|G|,(k,l,m,n)]$, there are at most three possible geometrically distinct angle sequences. These three types can be denoted (k,l,m,n), (k,m,l,n), and (k,l,n,m) quadrilaterals. If a group G gives a geodesic tiling with one of these tiles, it gives a geodesic tiling with the other two tiles. If an Abelian group A gives a geodesic, kaleidoscopic tiling with one of these tiles, it gives a geodesic, kaleidoscopic tiling with the other two tiles.*

Proof The proof follows from the definitions and the preceding discussion. The last statement follows from Proposition 7. ■

Example 9 *For distinct integers $k,l,m \geq 2$ and $n = lcm(k,l,m)$, [5] shows there exists a* *geodesic, kaleidoscopic tiling by (k,l,m,n), (k,m,l,n), and (k,l,n,m) quadrilaterals with the cyclic group of order n. In [7], Maclachlan gives necessary and sufficient conditions for the existence of geodesic, kaleidoscopic tilings by Abelian groups. These necessitate the existence of geodesic, kaleidoscopic tilings with the above quadrilaterals with Abelian groups of orders kn, ln, and mn. Note that all of these groups are Abelian and thus the same genus surface is tiled with each of the three tiles.*

5.3 Extension to polygons

Another type of tiling of particular interest is those by equi-angular polygons. We will describe those in the next subsection. In this subsection we will derive the criteria for constructing kaleidoscopic tilings by an arbitrary kaleidoscopic polygon.

Before proceeding we need to develop the analogues of the criteria (5) for the tiling to be kaleidoscopic. To this end, let the vertices of the master polygon be $P_1, \ldots, P_n = P_0$ and let p_j denote both the edge and the reflection in the side $\overline{P_{j-1}P_j}$, where subscripts are taken mod n. Further, let

$$a_j \quad = \quad p_j p_{j+1}, \tag{10}$$

so that

$$p_{j+1} \quad = \quad a_j^{-1} p_j. \tag{11}$$

Also define

$$\theta_j : G \to G, \ \ \theta_j(g) = p_j g p_j.$$

With this terminology, a_j is the stabilizer at P_j and θ_j is reflection in p_j. The action of θ_1 on a_j is given by:

$$\begin{aligned} \theta_1(a_j) &= p_1(p_j p_{j+1})p_1 \\ &= p_1(p_2p_2)\cdots(p_{j-1}p_{j-1})(p_jp_{j+1})(p_jp_j)\cdots(p_2p_2)p_1 \\ &= (p_1p_2)(p_2p_3)\cdots(p_{j-1}p_j)(p_{j+1}p_j)(p_jp_{j-1})\cdots(p_3p_2)(p_2p_1) \\ &= (a_1a_2\cdots a_{j-1})a_j^{-1}(a_{j-1}^{-1}\cdots a_2^{-1}a_1^{-1}). \end{aligned}$$

or upon setting

$$b_j = a_1 a_2 \cdots a_j, \tag{12}$$

we get

$$\theta_1(a_j) \quad = \quad b_{j-1} a_j^{-1} b_{j-1}^{-1}. \tag{13}$$

Finally (11) implies that $p_k = b_{k-1}^{-1} p_1$ and hence

$$\theta_k(a_j) \quad = \quad b_{k-1}^{-1} b_{j-1} a_j^{-1} b_{j-1}^{-1} b_{k-1}. \tag{14}$$

Remark 5 *With the appropriate identifications, the automorphism θ of (5) becomes θ_2 in our current setup.*

The criteria for a tiling may be stated in our current terminology:

Theorem 10 *Let G be a group. Suppose that $(k_1, k_2, \ldots, k_n)$ is an n-tuple of integers, all greater than 1, all dividing $|G|$, and such that*

$$\sigma = 1 + \frac{|G|}{2}\left(n - 2 - \sum_{j=1}^{n} \frac{1}{k_j}\right) \tag{15}$$

is an integer. Suppose further that $(a_1, a_2, \ldots, a_n)$ is an n-tuple of elements of G such that

$$\begin{aligned} &o(a_j) = k_j \\ &a_1 a_2 \cdots a_n = 1 \\ &G = \langle a_1, a_2, \ldots, a_n \rangle, \end{aligned} \tag{16}$$

and that Ω_0 is any $(k_1, k_2, \ldots, k_n)$-polygon. Then G is (isomorphic to) a conformal tiling group of a geodesic, locally kaleidoscopic tiling of a surface S of genus σ by $(k_1, k_2, \ldots, k_n)$-polygons, congruent to Ω_0, though the tiling need not be kaleidoscopic. The tiling may be constructed so that the n-tuple $(a_1, a_2, \ldots, a_n)$ are the rotations at the vertices of some appropriately chosen quadrilateral, with appropriate ordering. Furthermore, if $(a'_1, a'_2, \ldots, a'_n) = \omega \cdot a'_1, a'_2, \ldots, a'_n)$ for some $\omega \in Aut(G)$ then the surface S' constructed from $(a'_1, a'_2, \ldots, a'_n)$ and Ω_0 is conformally equivalent to S except that the G-action on S' is twisted by ω.

The kaleidoscopic conditions are stated in the following theorem.

Theorem 11 *Let G be a group, and $(k_1, k_2, \ldots, k_n)$, $(a_1, a_2, \ldots, a_n)$ be as in Theorem 10 and let b_j be as defined in (12). Then the tiling on S, constructed in Theorem 10 is kaleidoscopic if and only if there is an involutary automorphism $\theta \in Aut(G)$ such that*

$$\theta(a_j) \quad = \quad b_{j-1} a_j^{-1} b_{j-1}^{-1}, 1 \leq j \leq n, \tag{17}$$

or alternatively

$$\theta(b_j) \quad = \quad b_j^{-1}, 1 \leq j \leq n. \tag{18}$$

Proof We just sketch a proof. The reflections in the sides of Ω_0 generate a tiling of the hyperbolic plane $\mathbb{H}$ by isometric images of Ω_0. Let Λ^* and Λ be the full and conformal tiling groups respectively. The conditions on G indicate that here is an epimorphism $\eta : \Lambda \to G$.

Let Γ be the kernel of this action. Then Γ has a fixed-point-free action on $\mathbb{H}$ and the constructed tiling is Γ-equivariant. Thus $S = \mathbb{H}/\Gamma$ has a tiling by polygons congruent to Ω_0 and $G = \Lambda/\Gamma$ is the conformal tiling groups. If the automorphism θ exists then $\Gamma \lhd \Lambda^*$ and the full tiling group acts on S.

The two sets of equations (17) and (18) are easily proven to be equivalent upon noting that $a_j = b_{j-1}^{-1} b_j$. Note that if any one of the θ_k are automorphisms then they all are. The most convenient form can be used for computation. ■

5.4 Equiangular tilings

This subsection gives several infinite classes of tilings with equi-angular polygons, using the criteria of the last section.

Proposition 12 *Define $G_i = \mathbb{Z}_a^i$. For any integer $a \geq 3$, there exists a geodesic, kaleidoscopic tiling by $(a, a, \ldots, a)$ n-gons ($n \geq 3$) with conformal tiling group G_i for any i, $1 \leq i \leq n-1$. If $i = 1$ and a is even, then n must be even.*

Proof The existence of the tilings is shown by conditions in [7], but $n-1$ tiling groups for a given equiangular n-gon are specified here. The genus of the surface will be

$$\sigma = \frac{1}{2} a^{i-1}(n(a-1) - 2a) + 1.$$

according to the Riemann-Hurwitz equation, thus the parity conditions on a and n are necessary for $i = 1$. We first need to construct n generators of order a in G_i. In $\mathbb{Z}_a$ we are guaranteed an identity e and element α, $o(\alpha) = a$. Assume first that $i > 1$ We define the first i generators $a_j, 1 \leq j \leq i$ by

$$a_j = (x_1, \ldots, x_i),\ x_j = \alpha,\ x_k = e, k \neq j.$$

If $n - i$ is odd, the remaining elements are

$$a_{i+1} = (\alpha^{-1}, \alpha^{-1}, \ldots, \alpha^{-1})$$

and $(n - i - 1)/2$ pairs of elements of the form

$$(\alpha^{-1}, e, e, \ldots, e),\ (\alpha, e, e, \ldots, e). \tag{19}$$

If $n - i$ is even the remaining elements are

$$a_{i+1} = (\alpha^{-1}, e, e, \ldots, e),\quad a_{i+2} = (e, \alpha^{-1}, \alpha^{-1}, \ldots, \alpha^{-1})$$

and $(n-i-2)/2$ pairs of elements as given in (19). If $i=1$ and n is even then generators may be constructed as $n/2$ pairs of α and α^{-1}. If a is odd and n is odd then set

$$a_1 = \alpha^{-1},\ a_2 = \alpha^{-1},\ a_3 = \alpha^2,$$

and the remaining generators are $(n-3)/2$ pairs of α and α^{-1}. It is clear that in every case the n constructed elements generate G_i, are each of order a, and that their product equals $(e, e, \ldots, e)$. Since G is Abelian, the automorphism

$$\theta(x) = x^{-1},\ \forall x \in G_i$$

fulfills the requirements for G_i to geodesically, kaleidoscopically tile a surface of the given genus. ■

The following proposition does not generalize to n-gons but does delve into the realm of non-Abelian conformal tiling groups.

Proposition 13 *For an integer a, define A to be the $(b-1)\times(b-1)$ matrix of the form*

$$\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 1 & 0 \\ 0 & \ldots & 0 & 1 & 1 \end{bmatrix}.$$

Then define

$$G_{a,b} = \mathbb{Z}_a \ltimes \mathbb{Z}_a^{b-1} = \langle x, y_i : x^a = y_i^a = [y_i, y_j] = e,\ y_i^x = Ay_i,\ 1 \le i, j \le b-1 \rangle,$$

where y_i has 1 in location i, and zero elsewhere. If $b \ge 3$, and a is an integer $a \ge b-1$, and $\gcd(a, (b-2)!) = 1$ then there exists a geodesic tiling by (a, a, a, a) quadrilaterals by the non-Abelian conformal tiling group $G_{a,b}$, of order a^b.

Proof Note that the above presentation of $G_{a,b}$ yields the following group relations:

$$y_i = y_{i-1}^{-1} y_{i-1}^x \text{ for } 1 < i \le b-1 \text{ and} \qquad (20)$$
$$z^x = Az,\ \forall z \in \mathbb{Z}_a^{b-1} \subset G_{a,b}.$$

We need the conditions on a in order to guarantee that the map induced on $\mathbb{Z}_a^{b-1}$ by conjugation by x has order a. For, note that $A = I + J$ where I is the identity matrix and

J is nilpotent of order $b-1$. Then

$$A^a = (I+J)^a = I + \sum_{k=1}^{b-2} \binom{a}{k} J^k = I \tag{21}$$

since $\binom{a}{k} = a(a-1)\cdots(a-k-1)/k!$, then $\binom{a}{k} \equiv 0 \bmod a$. Since x and y_1 have order a, the quadruple of elements $(x, x^{-1}, y_1, y_1^{-1})$ has the desired orders. That the product of the elements is the identity is trivial, so the only remaining condition for the existence of geodesic tiling is that these elements generate $G_{a,b}$. Define

$$H = \langle x, x^{-1}, y_1, y_1^{-1} \rangle.$$

We know $x, y_1 \in H$. Assume $y_i \in H$, then due to (20) and induction, $y_{i+1} \in H$ for $1 \leq i < b-1$. Thus all generators are in the group and $H = G_{a,b}$. ■

Conjecture 14 *The geodesic tilings in the above proposition are also kaleidoscopic.*

Ideas for proving this conjecture are given in Section 6.

For our next example we define, for any odd prime p and positive integer i,

$$\begin{aligned} G_i \;=\; & \langle x_1, x_2, \ldots, x_i, y_1, y_2, \ldots, y_i, z : {x_j}^p = {y_j}^p = z^p = [x_j, z] = [y_j, z] = 1, 1 \leq j \leq i, \\ & [x_j, x_k] = [y_j, y_k] = 1, 1 \leq j, k \leq i, \\ & [x_j, y_k] = 1, 1 \leq j \neq k \leq i, [x_j, y_j] = z, 1 \leq j \leq i \rangle. \end{aligned}$$

Then G_i is an extra-special group of order p^{2i+1}, and thus, ζG_i, the center of the group, equals $[G_i, G_i] = \langle z \rangle$. Explanations of several desired properties of such groups can be found in [8]. Two in particular are the following. First let $H_m = \langle x_m, y_m \rangle$. Then, if $m \neq n$ $[H_m, H_n] = 1$, so the elements of H_m commute with those of H_n. Second we have:

$$\Gamma(\zeta G_i) = \zeta G_i, \;\; \forall \Gamma \in \mathrm{Aut}(G_i).$$

These properties will help us easily prove the following proposition.

Proposition 15 *Let G_i be defined as above. Suppose that a partially defined mapping $\phi : \{z, x_j, y_j : 1 \leq j \leq i\} \to G_i$ satisfies $\phi(z) = z$, and for each j, $1 \leq j \leq i$, satisfies one of the three conditions below:*

$$\phi(x_j) = {x_j}^{-1}, \;\; \phi(y_j) = {y_j}^{-1} \quad \textit{or} \tag{22}$$

$$\phi(x_j) = {x_j}^{-1}, \;\; \phi(y_j) = x_j {y_j}^{-1} {x_j}^{-1} \quad \textit{or} \tag{23}$$

$$\phi(x_j) = {y_j}^{-1} {x_j}^{-1} y_j, \;\; \phi(y_j) = {y_j}^{-1}. \tag{24}$$

Then ϕ extends to an automorphism of G_i.

Proof If the relations in the presentation hold after applying ϕ, it is an automorphism. Thus, we first need to show that all elements are mapped to elements of the same order:

$$(x_j{}^{-1})^p = (y_j{}^{-1})^p = (x_j y_j{}^{-1} x_j{}^{-1})^p = (y_j{}^{-1} x_j{}^{-1} y_j)^p = z^p = 1.$$

The non-trivial of these equalities are

$$(x_j y_j{}^{-1} x_j{}^{-1})^p = x_j y_j{}^{-p} x_j{}^{-1} = 1$$

and

$$(y_j^{-1} x_j{}^{-1} y_j)^p = y_j^{-1} x_j{}^{-p} y_j = 1.$$

Next we need to show that the commutation relation defining z holds, i.e., $[\phi(x_j), \phi(y_j)] = \phi(z) = z$. In the three different cases we have:

$$\begin{aligned}
[\phi(x_j), \phi(y_j)] &= [x_j^{-1}, y_j^{-1}] = [x_j, y_j]^{y_j x_j} = z, \\
[\phi(x_j), \phi(y_j)] &= [x_j^{-1}, x_j y_j{}^{-1} x_j{}^{-1}] = [x_j, y_j]^{y_j} = z, \\
[\phi(x_j), \phi(y_j)] &= [y_j^{-1} x_j^{-1} y_j, y_j^{-1}] = [x_j, y_j]^{y_j x_j y_j} = z.
\end{aligned}$$

Next it immediately follows that all commutation relations involving z hold. Finally note that $\phi(H_m) = H_m$ and hence the relations that show $[H_m, H_n] = 1$ for $m \neq n$ also hold. We have shown that every relation is preserved. ■

Proposition 16 *There exists a geodesic, kaleidoscopic tiling with conformal tiling group G_i as defined above with equi-angular n-gons (n is even) of the form (p,p,...) for any odd prime p and for each i, where*

$$\begin{aligned}
1 \leq i \leq \frac{n}{4}, \quad if \quad n \equiv 0 \ mod \ 4, \\
1 \leq i \leq \frac{n-2}{4}, \quad if \quad n \equiv 2 \ mod \ 4.
\end{aligned}$$

Proof We consider the two cases separately:
Case $n \equiv 0$ mod 4. First let us assume $i = n/4$. We claim that the n-tuple of elements of G_i,

$$(x_1, y_1, y_1^{-1}, x_1^{-1}, x_2, y_2, y_2^{-1}, x_2^{-1}, \ldots, x_i, y_i, y_i^{-1}, x_i^{-1}) \tag{25}$$

satisfies the properties required to construct a G_i-kaleidoscopic tiling by $(p, p, \ldots, p)$-gons. Let a_j be the j^{th} element of the above n-tuple. That $o(a_j) = p$ and that $\langle a_j | 1 \leq j \leq n \rangle = G_i$ follow directly from the group presentation. That $a_1 a_2 \ldots a_n = 1$ is also trivial. Now we shall apply the criterion (18). The sequence of b_j's are

$$(x_1, x_1y_1, x_1, 1, x_2, x_2y_2, x_2, 1, \ldots, x_i, x_iy_i, x_i, 1),$$

By the criterion, the θ-images of the b_j's, are the inverses of the b_j's:

$$(x_1^{-1}, y_1^{-1}x_1^{-1}, x_1^{-1}, 1, x_2^{-1}, y_2^{-1}x_2^{-1}, x_2, 1, \ldots, x_i^{-1}, y_i^{-1}x_i^{-1}, x_i^{-1}, 1).$$

Therefore, we need θ to satisfy: $\theta(x_j) = x_j^{-1}$ and $y_j^{-1}x_j^{-1} = \theta(x_jy_j) = x_j^{-1}\theta(y_j)$. It follows that we should choose the automorphism defined by equation (23). Thus, G_i geodesically and kaleidoscopically tiles the $(p, p, \ldots, p)$ n-gons. The cases with $i < n/4$ can be proven in a similar manner by using a n-tuple of the form

$$(x_1, y_1, y_1^{-1}, x_1^{-1}, x_2, y_2, y_2^{-1}, x_2^{-1}, \ldots, x_i, y_i, y_i^{-1}, x_i^{-1}, x_i, y_i, y_i^{-1}, x_i^{-1}, \ldots).$$

Case $n \equiv 2 \mod 4$. Here we simply prepend x_1, x_1^{-1} to the beginning of the sequence given in (25). ■

Remark 6 *There is no indication that a geodesic, kaleidoscopic tiling by G_i for some i with equiangular n-gons does not exist when n is odd. However, note that the existence of geodesic, kaleidoscopic tilings by G_1 and (p, p, p) triangles is given in [10].*

In Appendix A, the tables of tiling classifications show that, at least for low genus, there exist very few geodesic tilings that are not kaleidoscopic. The following proposition and conjecture give one possible class of geodesic, non-kaleidoscopic tilings.

Proposition 17 *Let x, y_1, and y_2 be generators of $\mathbb{Z}_2$, $\mathbb{Z}_a$, and $\mathbb{Z}_9$ respectively. Define*

$$\begin{aligned} G_a &= \mathbb{Z}_2 \ltimes (\mathbb{Z}_a \times \mathbb{Z}_9), \text{ where} \\ y_1^x &= y_1^{a-1} = y_1^{-1} \text{ and} \\ y_2^x &= y_2^8 = y_2^{-1}. \end{aligned}$$

For a positive integer a, not divisible by three, there exists geodesic tilings with branching data $[18a, (2, 2, 3, 9a)]$ with conformal tiling group G_a.

Proof Consider the quadruple

$$(x, xy_2^2y_1^{a-1}, y_2^6, y_2y_1).$$

That $o(x) = 2$ and $o(y_2^6) = 3$ is clear, as is $o(y_2y_1) = 9a$ if we note that y_1 and y_2 commute and that a and nine are relatively prime. The second element has order two because

$$(xy_2^2y_1^{a-1})^2 = xy_2^2y_1^{a-1}xy_2^2y_1^{a-1}$$

$$
\begin{aligned}
&= (xx)(xy_2^2y_1^{a-1}x)y_2^2y_1^{a-1} \\
&= (xx)((y_2^x)^2(y_1^x)^{a-1}y_2^2y_1^{a-1} \\
&= y_2^{-2}y_1^{1-a}y_2^2y_1^{a-1}
\end{aligned}
$$

The product of the four elements is the identity because

$$
\begin{aligned}
(x)(xy_2^2y_1^{-1})(y_2^6)(y_2y_1) &= xxy_2^2y_1^{a-1}y_2^6y_2y_1 \\
&= x^2y_2^9y_1^a = e.
\end{aligned}
$$

The first and last elements of the quadruple generate G_a, which is clear once the following is realized:

$$
\begin{aligned}
(y_2y_1)^a &= y_2^a \textit{ and } o(y_2^a) = 9, \textit{ and} \\
(y_2y_1)^9 &= y_1^9 \textit{ and } o(y_1^9) = a.
\end{aligned}
$$

Thus, by Proposition 8, G_a is a conformal tiling group for two different geodesic tilings, one with $(2, 2, 3, 9a)$-quadrilateral tiles and the other with $(2, 3, 2, 9a)$ tiles. ■

Conjecture 18 *The tiling with* $(2, 2, 3, 9a)$ *quadrilaterals described above is kaleidoscopic, while the tiling with* $(2, 3, 2, 9a)$ *quadrilaterals is non-kaleidoscopic.*

6 Future work

Here is some unfinished work

- Complete the classification of tilings by pentagons, etc. for low genus.
- Find proofs of the conjectures in section 5.

The following are ideas for proving Conjecture 14. One method of proving that the tilings are also kaleidoscopic would be to calculate the necessary conditions for the desired automorphism for the given general quadruple and then prove that the relations in the groups presentation still hold after applying these conditions. This method does not seem to form a clean proof. Perhaps a better way would be to represent the group $G_{a,b}$ as a group of matrices because then the automorphism would be conjugation by a matrix for which a general form can be found. An example of this group representation for $b = 4$ is

$$
G_{a,4} = \left\{ \begin{bmatrix} 1 & w & 0 & x \\ 0 & 1 & w & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix} : w, x, y, z \in \mathbb{Z}_a \right\}.
$$

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Appendix A. Classification tables of geodesic tilings for genus 2-5

Note: Tables including all geodesic tilings for genus 2-12 can be found at [13].

G	$\|G\|$	Quadrilateral	Presentation	Tiling Type
$\mathbb{Z}_3$	3	(3^4)		Kaleidoscopic
$\mathbb{Z}_4$	4	$(2^2, 4^2)$ $(2, 4, 2, 4)$		Kaleidoscopic Kaleidoscopic
$\mathbb{Z}_6$	6	$(2^2, 3^2)$ $(2, 3, 2, 3)$		Kaleidoscopic Kaleidoscopic
$\mathbb{D}_3$	6	$(2^2, 3^2)$ $(2, 3, 2, 3)$		Kaleidoscopic Non-Kaleidoscopic
$\mathbb{D}_4$	8	$(2^3, 4)$		Kaleidoscopic
$\mathbb{Z}_2 \times \mathbb{D}_3$	12	$(2^3, 4)$		Kaleidoscopic

Table 1: Genus 2

G	$\lvert G\rvert$	Quadrilateral	Presentation	Tiling Type
$\mathbb{Z}_4$	4	(4^4)		Kaleidoscopic
$\mathbb{Z}_6$	6	$(2, 3^2, 6)$		Kaleidoscopic
		$(2, 3, 6, 3)$		Kaleidoscopic
		$(2^2, 6^2)$		Kaleidoscopic
		$(2, 6, 2, 6)$		Kaleidoscopic
$\mathbb{D}_4$	8	$(2^2, 4^2)$		Kaleidoscopic
		$(2, 4, 2, 4)$		Non-Kaleidoscopic
$\mathbb{Z}_2 \times \mathbb{Z}_4$	8	$(2^2, 4^2)$		Kaleidoscopic
		$(2, 4, 2, 4)$		Kaleidoscopic
$\mathbb{A}_4$	12	$(2^2, 3^2)$		Kaleidoscopic
		(2,3,2,3)		Kaleidoscopic
$\mathbb{Z}_2 \times \mathbb{D}_3$	12	$(2^3, 6)$		Kaleidoscopic
$\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_4)$	16	$(2^3, 4)$	$\left\langle x, y : y^x = \left\lvert \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{smallmatrix} \right\rvert \times y \right\rangle$	Kaleidoscopic
$\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$	16	$(2^3, 4)$	$\left\langle x, y : y^x = \left\lvert \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{smallmatrix} \right\rvert \times y \right\rangle$	Kaleidoscopic
Σ_4	24	$(2^3, 3)$		Kaleidoscopic

Table 2: Genus 3

G	$\|G\|$	Quadrilateral	Presentation	Tiling Type
$\mathbb{Z}_5$	5	(5^4)		Kaleidoscopic
$\mathbb{Z}_6$	6	$(2,6^3)$		Kaleidoscopic
	6	$(3^2,6^2)$		Kaleidoscopic
	6	$(3,6,3,6)$		Kaleidoscopic
$\mathbb{H}$	8	$(2,4^3)$		Kaleidoscopic
$\mathbb{Z}_8$	8	$(2^2,8^2)$		Kaleidoscopic
	8	$(2,8,2,8)$		Kaleidoscopic
$\mathbb{Z}_3 \times \mathbb{Z}_3$	9	(3^4)		Kaleidoscopic
$\mathbb{D}_5$	10	$(2^2,5^2)$		Kaleidoscopic
	10	$(2,5,2,5)$		Kaleidoscopic
$\mathbb{Z}_{10}$	10	$(2^2,5^2)$		Kaleidoscopic
	10	$(2,5,2,5)$		Kaleidoscopic
$\mathbb{A}_4$	12	$(2,3^3)$		Kaleidoscopic
$\mathbb{Z}_2 \times \mathbb{Z}_6$	12	$(2^2,3,6)$		Kaleidoscopic
	12	$(2,3,2,6)$		Kaleidoscopic

Table 3: Genus 4

G	$\lvert G \rvert$	Quadrilateral	Presentation	Tiling Type
$\mathbb{Z}_3 \times \mathbb{D}_3$	18	$(2^2, 3^2)$		Kaleidoscopic
	18	$(2, 3, 2, 3)$		Kaleidoscopic
$\mathbb{Z}_2 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_2)$	18	$(2^2, 3^2)$		Kaleidoscopic
	18	$(2, 3, 2, 3)$		Kaleidoscopic
$\mathbb{Z}_2 \times \mathbb{D}_5$	20	$(2, 2, 2, 5)$		Kaleidoscopic
$\mathbb{D}_6 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$	24	$(2^3, 4)$		Kaleidoscopic
	36	$(2^3, 3)$		Kaleidoscopic

Table 4: Genus 4 continued

G	$\lvert G\rvert$	Quadrilateral	Presentation	Tiling Type
	6	(6^4)	$\langle a_1, a_2 \mid a_1{}^2 = e, a_2{}^3 = e\rangle$	Kaleidoscopic
	8	(4^4)	$\langle a_1, a_2, a_3 \mid a_1{}^2 = a_3\rangle$	Kaleidoscopic
	8	(4^4)	$\langle a_1, a_2, a_3 \mid a_1{}^2 = a_3, a_2{}^2 = a_3, a_2{}^{a_1} = a_2a_3\rangle$	Kaleidoscopic
	8 8	$(2,4,8^2)$ $(2,8,4,8)$	$\langle a_1, a_2, a_3 \mid a_1{}^2 = a_2, a_2{}^2 = a_3\rangle$	Kaleidoscopic Kaleidoscopic
	10 10	$(2^2,10^2)$ $(2,10,2,10)$	$\langle a_1, a_2 \mid a_1{}^2 = e, a_2{}^5 = e\rangle$	Kaleidoscopic Kaleidoscopic
	12	(3^4)	$\langle a_1, a_2, a_3 \mid a_1{}^3 = e, a_2{}^2 = e,$ $a_3{}^2 = e, a_2{}^{a_1} = a_3, a_3{}^{a_1} = a_2a_3\rangle$	Kaleidoscopic Kaleidoscopic
	12 12	$(2^2,6^2)$ $(2,6,2,6)$	$\langle a_1, a_2, a_3 \mid a_1{}^2 = e,$ $a_2{}^2 = e, a_3{}^3 = e, a_3{}^{a_1} = a_3{}^2$	Kaleidoscopic Kaleidoscopic
	12 12	$(2^2,6^2)$ $(2,6,2,6)$	$\langle a_1, a_2, a_3 \mid a_1{}^2 = e, a_2{}^2 = e, a_3{}^3 = e\rangle$	Kaleidoscopic Kaleidoscopic
	12 12	$(2,3,4^2)$ $(2,4,3,4)$	$\langle a_1, a_2, a_3 \mid a_1{}^2 = a_2,$ $a_2{}^2 = e, a_3{}^3 = e, a_3{}^{a_1} = a_3{}^2\rangle$	Kaleidoscopic Kaleidoscopic
	16 16	$(2^2,4^2)$ $(2,4,2,4)$	$\langle a_1, a_2, a_3, a_4 \mid a_1{}^2 = a_4, a_2{}^{a_1} = a_2a_3\rangle$	Kaleidoscopic Kaleidoscopic
	16 16	$(2^2,4^2)$ $(2,4,2,4)$	$\langle a_1, a_2, a_3, a_4 \mid a_1{}^2 = a_4, a_3{}^2 = a_4,$ $a_2{}^{a_1} = a_2a_3, a_3{}^{a_1} = a_3a_4, a_3{}^{a_2} = a_3a_4\rangle$	Kaleidoscopic Non-Kaleidoscopic
	16 16	$(2^2,4^2)$ $(2,4,2,4)$	$\langle a_1, a_2, a_3, a_4 \mid a_1{}^2 = a_4\rangle$	Kaleidoscopic Kaleidoscopic
	16 16	$(2^2,4^2)$ $(2,4,2,4)$	$\langle a_1, a_2, a_3, a_4 \mid a_2{}^{a_1} = a_2a_4\rangle$	Kaleidoscopic Kaleidoscopic

Table 5: Genus 5

G	$\lvert G \rvert$	Quadrilateral	Presentation	Tiling Type
	16 16	$(2^2, 4^2)$ $(2,4,2,4)$	$\langle a_1, a_2, a_3, a_4 \mid a_3{}^2 = a_4, a_2{}^{a_1} = a_2 a_4 \rangle$	Kaleidoscopic Kaleidoscopic
	20	$(2^3, 10)$	$\langle a_1, a_2, a_3, a_4 \mid a_1{}^2 = e,$ $a_2{}^2 = e, a_3{}^5 = e, a_3{}^{a_1} = a_3{}^4 \rangle$	Kaleidoscopic Kaleidoscopic
	24	$(2^3, 6)$	$\langle a_1, a_2, a_3, a_4 \mid a_1{}^2 = e, a_2{}^2 = e,$ $a_3{}^2 = e, a_4{}^3 = e, a_2{}^{a_1} = a_2 a_3, a_4{}^{a_1} = a_4{}^2 \rangle$	Kaleidoscopic Kaleidoscopic
	24	$(2^3, 6)$	$\langle a_1, a_2, a_3, a_4 \mid a_1{}^2 = e,$ $a_2{}^2 = e, a_3{}^2 = e, a_4{}^3 = e, a_4{}^{a_1} = a_4{}^2, \rangle$	Kaleidoscopic Kaleidoscopic
	24 24	$(2^2, 3^2)$ $(2,3,2,3)$	$\langle a_1, a_2, a_3, a_4 \mid a_1{}^2 = e,$ $a_2{}^3 = e, a_3{}^2 = e, a_4{}^2 = e, a_2{}^{a_1} = a_2{}^2,$ $a_3{}^{a_1} = a_4, a_3{}^{a_2} = a_4, a_4{}^{a_1} = a_3, a_4{}^{a_2} = a_3 a_4 \rangle$	Kaleidoscopic Kaleidoscopic
	24 24	$(2^2, 3^2)$ $(2,3,2,3)$	$\langle a_1, a_2, a_3, a_4 \mid a_1{}^2 = e, a_2{}^3 = e, a_3{}^2 = e,$ $a_4{}^2 = e, a_3{}^{a_2} = a_4, a_4{}^{a_2} = a_3 a_4 \rangle$	Kaleidoscopic Kaleidoscopic
	32	$(2^3, 4)$	$\langle a_1, a_2, a_3, a_4, a_5 \mid a_2{}^{a_1} = a_2 a_4, a_3{}^{a_1} = a_3 a_5 \rangle$	Kaleidoscopic
	32	$(2^3, 4)$	$\langle a_1, a_2, a_3, a_4, a_5 \mid a_2{}^2 = a_4,$ $a_2{}^{a_1} = a_2 a_4, a_3{}^{a_1} = a_3 a_5 \rangle$	Kaleidoscopic
	32	$(2^3, 4)$	$\langle a_1, a_2, a_3, a_4, a_5 \mid a_4{}^2 = a_5, a_2{}^{a_1} = a_2 a_4,$ $a_3{}^{a_1} = a_3 a_5, a_4{}^{a_1} = a_4 a_5, a_4{}^{a_2} = a_4 a_5 \rangle$	Kaleidoscopic
	48	$(2^3, 3)$	$\langle a_1, a_2, a_3, a_4 \mid a_1{}^2 = e, a_2{}^2 = e,$ $a_3{}^3 = e, a_4{}^2 = e, a_5{}^2 = e, a_3{}^{a_1} = a_3{}^2,$ $a_4{}^{a_1} = a_5, a_4{}^{a_3} = a_5, a_5{}^{a_1} = a_4, a_5{}^{a_3} = a_4 a_5 \rangle$	Kaleidoscopic

Table 6: Genus 5 continued