separability of tilings

jason deblois* — university of chicago

nick baeth* — pacific lutheran university

lisa powell* — harvard university

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Abstract

A tiling by triangles of an orientable surfaces is called *kaleidoscopic* if the local reflection in any edge of a triangle extends to a global isometry of the surface. Given such a global reflection the fixed point subset of the reflection consists of embedded circles (*ovals*) whose union is called the *mirror* of the reflection. The reflection is called *separating* if removal of the mirror disconnects the surface into two components. We consider surfaces such that the orientation preserving subgroup of the tiling group generated by the reflections is cyclic or abelian. A complete classification of those surfaces with separating reflections is obtained in the cyclic case as well as partial results for abelian, non-cyclic groups.

Contents

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1 Introduction

Consider a surface S with a geodesic, kaleidoscopic tiling by triangles (defined more precisely in a later section). Examples of tilings are the icosahedral tiling of the sphere in Figure 1 and the tiling of the torus in Figure 2. Associated to these tilings are triples of integers (k, l, m) which describe the angles $\frac{\pi}{k}$, $\frac{\pi}{l}$, and $\frac{\pi}{m}$ of a triangular tile. The icosahedral tiling in Figure 1 is a $(2, 3, 5)$-tiling and the toral tiling in Figure 2 is a $(2, 4, 4)$-tiling. Kaleidoscopic tilings have rotational symmetry, again as explained in a later section. For the time being, imagine rotating the sphere at vertices so as to return the tiling to its original position, just as one would rotate a soccer ball to preserve its pattern. There is a (k, l, m)-triple of rotations $[a, b, c]$ at the vertices of tiles, through angles $\frac{\pi}{k}$, $\frac{\pi}{l}$, and $\frac{\pi}{m}$, of orders k, l, and m respectively. In particular, the triple of rotations satisfies $a^k = b^l = c^m = 1$. The triple $[a, b, c]$ generates a group G of rotations of S that we shall call the *conformal tiling group*, since it consists of orientation-preserving transformations. We call $[a, b, c]$ a generating (k, l, m)-triple of G. In this paper we will only consider tilings where the conformal tiling group is abelian. In the case of the sphere the conformal tiling group is A_5 so there are 60 rotations, including the identity.

In addition to having rotational symmetries the tiled surfaces also have reflectional symmetry. In the case of the icosahedral tiling the great circles defined by the tiling are *mirrors* of the reflectional symmetry. By cutting along the mirror of reflectional symmetry the sphere falls into two identical halves and hence the reflection is called *separating reflection* or the mirror is called a *splitting mirror*. On the torus, the splitting mirrors are a bit less easy to see, the "horizontal" and "vertical" mirrors each have two components (illustrated by cutting a bagel in half in either of the two canonical ways). However the inclined mirrors have only one component and do not split the surface. (Note that we have to embed the surface in a higher dimensional space to realize the inclined reflections as isometries. Naturally, more intricate examples of tilings and splitting lay in surfaces of higher genus than the sphere and the torus.

Our research has two primary goals:

Goal 1: To classify splitting properties for surfaces for which the conformal tiling group is cyclic.

Goal 2: To formulate splitting properties for surfaces with abelian conformal tiling groups.

We also became interested in the structure of the tiling groups, related to the classification of tilings up to isometry. We have partial results which translates into group theory as:

Goal 3: To examine the automorphism classes of generating (k, l, m)-triples $[a, b, c]$ triples when (k, l, m) is of the form (k, dk, dk), $d \geq 1$.

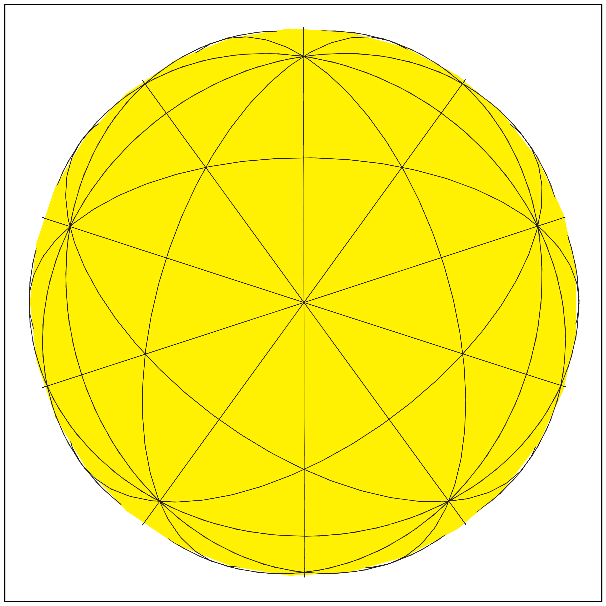

Figure 1. Icosahedral tiling of the sphere.

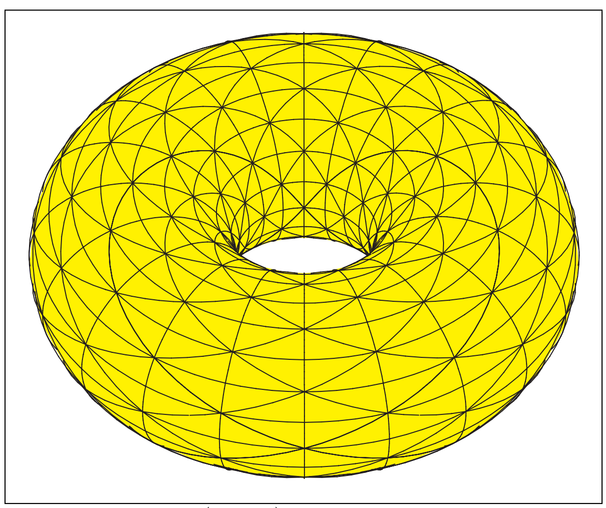

Figure 2. $(2, 4, 4)$ tiling of the torus.

The main contributions of this paper are the complete classification of splitting properties of tilings constructed from a cyclic group, the number of automorphism classes of a particular type of tilings constructed from cyclic groups, and partial results on the splitting properties of tilings constructed from abelian groups. This paper extends the work of Belk [2] in particular we make strong use of the Reflective Walk Algorithm developed in that paper. Other works in which separability properties of families of surfaces are discussed are [6] and [7], indeed we utilize some fixed point formulas from [6].

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2 Background on surfaces and tilings

The background material follows notes by S. Allen Broughton for his REU at the Rose-Hulman Institute of Technology [3]. Also consult background reference [1] for background on hyperbolic geometry.

Tilings of Surfaces A *surface* S is a compact orientable two-dimensional manifold, i.e., it looks locally like the plane. A surface may be represented topologically as a sphere with attached handles. The genus σ of a surface is the number of attached handles. We will be primarily concerned with hyperbolic surfaces, i.e., surfaces which have a metric of constant negative curvature and are geometrically modeled by the hyperbolic plane, described below. The genus must satisfy $\sigma \geq 2$. A *tiling* of a surface S is a non-overlapping covering of the surface by polygons, called tiles, where a polygon is a simply connected region bounded by a finite number of curves called edges. These edges meet in vertices at definite non-zero angles. We assume that two intersecting tiles will meet either along an edge or in a vertex. The tilings which we examine are *geodesic, kaleidoscopic* tilings:

- **Kaleidoscopic Condition** A tiling is kaleidoscopic if every edge e of the tiling is part of a closed curve, called a *geodesic*, on the surface such that there is a *mirror reflection* R_e of the surface, fixing the geodesic, and mapping tiles to tiles.
- **Geodesic Condition** A kaleidoscopic tiling is *geodesic* if for each reflection in the side of a tile the set of fixed points of the reflection, called the *mirror of the reflection,* is itself a union of edges. In other words, each edge is part of a closed smooth curve which is a union of edges. Each mirror is a disjoint set of smooth, closed curves each diffeomorphic to a circle; these circles are called *ovals*.

An example of a geodesic, kaleidoscopic tiling is the icosahedral tiling of the sphere, shown in Figure 1. An example of a kaleidoscopic tiling that is not geodesic is the tiling of the plane by regular hexagons. Though much of the work on tilings extends readily to other polygons, we will restrict ourselves to triangles.

Geometry, universal covers, and fundamental regions All of the polygons that meet at a vertex have the same angle measure at that vertex because the angles are congruent. This angle measure is $\frac{2\pi}{z}$, where z is the number of polygons meeting at the vertex. Because of the geodesic condition, the number of polygons is even, i.e. $z = 2y$, so the angle measure is $\frac{\pi}{y}$. Therefore a triangle will have angles $\frac{\pi}{k}$, $\frac{\pi}{l}$ and $\frac{\pi}{m}$ for some integers k, l, m. We call such a triangle a (k, l, m)-triangle. There is a relationship between the angle sum $\pi(\frac{1}{l} + \frac{1}{m} + \frac{1}{n})$ and the genus of the surface:

Angle Sums

genus	geometry type	angle sum	$\frac{1}{l} + \frac{1}{m} + \frac{1}{n}$
0	spherical	$> \pi$	> 1
1	euclidean	$= \pi$	$= 1$
≥ 2	hyperbolic	$< \pi$	< 1

We shall use the disc model $\mathbb{H}$ of hyperbolic geometry. The points of $\mathbb{H}$ are the interior points of the unit disc in the complex plane. The hyperbolic lines are the intersections with $\mathbb{H}$ of circles or lines perpendicular to the boundary of $\mathbb{H}$. The angle between lines is the ordinary angle between curves.

The reason for bringing the hyperbolic plane into play is that it is often easier to think of a surface as an identification space of a fundamental region of a tiling in the hyperbolic plane. To make this concept better understood let us first look at the euclidean example of the (2,4,4) tiling of the torus. If we cut open the torus along a vertical oval and a horizontal oval and flatten out the result we get a plane pattern that we see in Figure 3. We may recover the torus by making left-right and top-bottom identifications. Also we are easily able to identify the ovals and mirrors and decide whether a reflection is separating or not. If we were to extend the pattern to the entire plane we would get a tiling of the euclidean plane $\mathbb{E}$ and a covering map to $\mathbb{E} \rightarrow T$ that takes tiles to tiles in a 1-1 fashion. Thus each tiling of the torus lifts to a tiling on the universal cover, and each surface tiling can be realized by taking a fundamental region or polygon in the plane and identifying sides. Similarly, each tiling of a surface of genus $\sigma \geq 2$ lifts to a tiling on its universal cover, except that we now have a tiling of hyperbolic space. In Figure 4 a partial tiling of the hyperbolic plane by $(3, 3, 4)$-tiles is given. Again we may construct surfaces by selecting an appropriate polygon built up from tiles and then making the necessary identifications. An example of such a fundamental region is given in Figure 6 and the discussion thereafter.

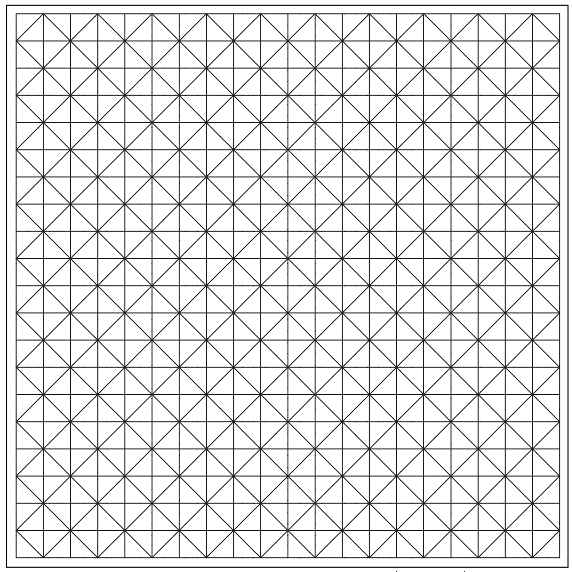

Figure 3. Fundamental region - $(2, 4, 4)$-tiling of torus

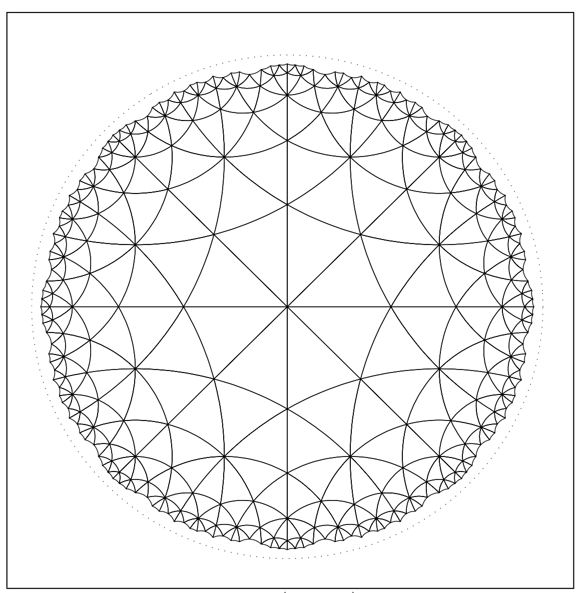

Figure 4. Partial $(3, 4, 4)$-tiling of $\mathbb{H}$

3 The tiling group, splitting and splitting tests

Tiling groups Each edge of the tiling determines a reflection, i.e. a transformation of the surface S onto itself. This transformation of the surface is an isometry, and it maps tiles to tiles. With these reflections we construct a group of symmetries G^* of the tiling, and use reflections in the edges of a tile to give a presentation of G^*. Select a master tile, Δ_0, displayed in Figure 5. We label the sides of Δ_0 and their corresponding reflections as p, q, r (here we have drawn the reflected images $p\Delta_0$ etc., with dotted lines). Since Δ_0 is a (k, l, m)-triangle, $a = pq$ is a counterclockwise non-euclidean rotation through $\frac{2\pi}{k}$ radians, yielding $q\Delta_0 \to p\Delta_0$. Similarly, $b = qr$ and $c = rp$ are counterclockwise rotations through $\frac{2\pi}{l}$ and $\frac{2\pi}{m}$ radians, respectively. Considering this and that reflections have order 2, we get:

$$o(a) = k,\ o(b) = l,\ o(c) = 1, \tag{1}$$

and

$$abc = pqqrrp = 1. \tag{2}$$

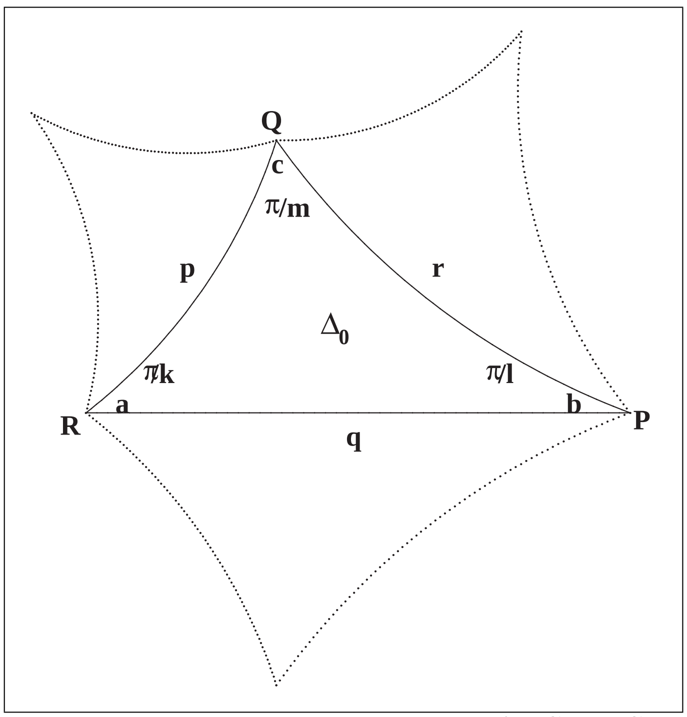

Figure 5. The master tile and generators for G and G^*

Let $G^* = \langle p, q, r\rangle$ and $G = \langle a, b, c\rangle = \langle a, b\rangle$ be the groups generated by the above elements. The subgroup $G \subseteq G^*$ is the subgroup of orientation-preserving or conformal isometries in G^*. Also, G is normal in G^* with index 2, and $G^* = \langle q \rangle \ltimes G$, a semi-direct product. We call G^* the *tiling group* of S and G the *orientation preserving (OP) tiling group* or the *conformal tiling group.* A triple of elements $[a, b, c]$ satisfying equations 1 and 2 is called a (k, l, m)-triple and is called a generating (k, l, m)-triple if and only if $G = \langle a, b, c \rangle$.

Conjugation of the generators a and b of G by q induces an automorphism θ satisfying:

$$\theta(a) = qaq^{-1} = qpqq = qp = a^{-1}$$
$$\theta(b) = qbq^{-1} = qqrq = rq = b^{-1}$$

The Riemann-Hurwitz equation gives the relationship between the order of the group G and the genus σ of the surface:

$$\frac{2\sigma - 2}{|G|} = 1 - \left(\frac{1}{k} + \frac{1}{l} + \frac{1}{m} \right).$$

The following proposition, a consequence of the Riemann existence theorem, allows us to construct surfaces directly from group theoretic constructions.

Proposition 1 *Let G have a generating (k, l, m)-triple, and suppose that the number σ defined by the Riemann-Hurwitz equation is an integer; then there is always a surface S of genus σ with an orientation preserving G-action. Additionally, if there is an involutary ($\theta^2 = id$) automorphism θ of G satisfying $\theta(a) = a^{-1}$ and $\theta(b) = b^{-1}$, then the surface S has a tiling T by (k, l, m)-triangles such that the conformal tiling group is the original G, and such that $G^* \simeq \langle \theta \rangle \ltimes G$.*

Proposition 2 *If G is abelian then $\theta : g \to g^{-1}$ satisfies the requirements of the involution in Proposition 1. Thus generating triples in abelian groups automatically define a tiling.*

Remark 3 *The formulae giving p, q, r in terms of θ and a, b, c are:*

$$p = \theta a^{-1} = a\theta$$
$$q = \theta$$
$$r = \theta b = b^{-1}\theta$$

We say that the tiling of S is induced by the triple (a, b, c).

Remark 4 *By the Poincaré polygon theorem [1, p. 242], the group G^* acts simply transitively on the tiles. Therefore, we can uniquely label the tiles of the surface with elements of G^*. More specifically, if Δ_0 is our master triangle and Δ' is any other triangle in the tiling there is a unique $g \in G^*$ such that $\Delta' = g\Delta_0$. Therefore the number of tiles on the surface is $|G^*| = 2\,|G|$.*

Notation 5 *We are denoting both edges and group elements by p, q, and r, a potentially confusing situation. If $g \in G^*$ and e is an edge then $g \cdot e$ will denote* *the edge e transformed by g if there is need to distinguish it from a group product. Thus $g \cdot p$, $g \cdot q$, and $g \cdot r$ will denote transformed edges and gp, gq, and gr will denote group products. The three edges $g \cdot p$, $g \cdot q$, and $g \cdot r$ bordering the tile $\Delta' = g\Delta_0$ will called the p-type, q-type, and r-type edges of Δ', respectively. Now observe that*

$$R_{g \cdot e} = gR_e g^{-1} \tag{3}$$

so that the reflections corresponding to $g \cdot p$, $g \cdot q$, and $g \cdot r$ are gpg^{-1}, gqg^{-1}, and grg^{-1}, respectively. Furthermore, if $e' = g \cdot e$ is a transformation of edges we may assume that $g \in G$. For if $g \in G^* - G$ then $g' = gR_e$ satisfies $g' \in G$, $e' = g' \cdot e$. Since R_e is the only element of G^* mapping e to itself, then, G permutes the sets of p-type edges simply transitively, and likewise for q-type and r-type edges.

Notation 6 *We adopt a similar notation and terminology for the vertices. The set of R-type vertices is $g_1 \cdot R, \ldots, g_s \cdot R$, where $G = g_1 \langle a \rangle \cup \cdots \cup g_s \langle a \rangle$ is the coset decomposition of G. Similar remarks apply to the P-type vertices and $\langle a \rangle$ and the Q-type vertices and $\langle c \rangle$. The order of vertex v is the order of its stabilizer: $|G_v| = |\{g \in G : gv = v\}|$.*

Notation 7 *Most of our triples will be selected from cyclic groups $G = \mathbb{Z}_n$ written additively. We will then use square brackets to denote a generating triple $[a, b, c]$ instead of parenthetical notation (a, b, c). For, there is great potential for confusion between triples of rotation orders (k, l, m) and triples of group elements with a far different interpretation.*

Equivalent tilings We use the notion of an isometry to define equivalent tilings. An isometry $T : X \to X'$ between two metric spaces (X, d) and (X', d') is a homeomorphism satisfying

$$d'(T(x), T(y)) = d(x, y), \forall x, y \in X.$$

where $d(x, y)$ is the distance between the points x and y. The isometries of surfaces are derived from the geometry of their universal covers, be it spherical, Euclidean or hyperbolic. The *conformal isometry group* $\mathrm{Aut}(S)$ of a surface is the group of conformal (angle-preserving or orientation-preserving) isometries. The group $\mathrm{Aut}^*(S)$ of isometries also includes, if any, the anti-conformal isometries (angle-reversing or orientation-reversing). For any surface, if $\mathrm{Aut}^*(S)$ contains any anti-conformal maps, then, $\mathrm{Aut}(S)$ is of index 2 in $\mathrm{Aut}^*(S)$. If the genus of S is 2 or greater then these groups are finite.

We say that an abstract group G^* acts as a tiling group of S if there is a monomorphism $\epsilon : G^* \to \mathrm{Aut}^*(S)$ defined by a kaleidoscopic, geodesic tiling of S. Note that $\mathrm{Aut}^*(S)$ may be strictly larger than G^*. Similar remarks apply for conformal tiling groups.

Definition 8 *Suppose that G^* acts on two surfaces S, S', defined by tilings on the surfaces. Then we say that the actions are* isometrically equivalent *if there* *is an* $h : S \to S'$ *and an automorphism of* G^* *such that*

$$h(g \cdot x) = \omega(g) \cdot h(x) \ \forall g \in G^*, x \in S.$$

If the tiling action of G^* on two surfaces is isometrically equivalent then the surfaces have identical geometries. For, the isometric equivalence maps tiles to tiles, and preserves all geometric data. Isometrically equivalent actions on the same surface amounts to a relabelling of the tiles by the elements of G^*.

Definition 9 *Two generating triples* $[a, b, c]$ *and* $[a', b', c']$ *are called* $\mathrm{Aut}(G)$*-equivalent if there is an automorphism* ω *of* G *such that* $a' = \omega(a), b' = \omega(b)$, *and* $c' = \omega(c)$. *If* $[a, b, c]$ *determines a tiling of a surface* S *as in Proposition 1 then* $[a', b', c']$ *determines an isometrically equivalent tiling of an* S' *with* $\theta' = \omega \circ \theta \circ \omega^{-1}$.

It is possible for $\mathrm{Aut}(G)$-inequivalent triples $[a, b, c]$ and $[a', b', c']$ to generate isometric tilings. A characterization of when generating triples $[a, b, c]$, $[a', b', c']$ generate isometrically equivalent tilings, for abelian conformal tiling groups, is given in the next Proposition. For the general case see [4].

Proposition 10 *Suppose that* $[a, b, c]$ *and* $[a', b', c']$ *are two* (k, l, m)*-generating triples for an abelian conformal tiling group* G. *The triples generate isometrically equivalent tilings if and only if there is a permutation where* π *of* $[a, b, c]$, *preserving the order of elements, and an automorphism* $\omega \in \mathrm{Aut}(G)$, *such that*

$$[a', b', c'] = \omega \cdot [\pi a, \pi b, \pi c]. \tag{4}$$

Remark 11 *In a similar vein, if* (k', l', m') *is a permutation of* (k, l, m) *there is a* $1 - 1$ *correspondence between generating triples and isometry classes of* (k, l, m)*-tilings and* (k', l', m')*-tilings. Thus we may assume that* $k \leq l \leq m$, *when we need to do so.*

Separating reflections and mirrors A reflection R of a surface is *separating* if by cutting along all of the ovals of its mirror $S_R = \{x \in S : R(x) = x\}$ we separate the surface into two pieces (cf. Proposition 12 to follow). Otherwise it is called *non-separating*. If a reflection is separating, we say that the tiling splits along the mirror S_R of that reflection, or more simply that S splits at R. For example, if we cut along a mirror of a reflection in great circle on a sphere the surface will separate into two hemispheres. The torus example is less intuitive. For example, the $(2, 4, 4)$ tiling of the torus splits along the p and q mirrors, but not along the r mirror, where r is the diagonal reflection. It will be helpful for later exposition to reproduce the well known proof that a reflection separates the surface into one or two pieces

Proposition 12 *Let* R *be a reflection of a surface* S *and* S_R *its mirror. Then, either* $S - S_R$ *is a path connected open subset of* S *or* $S - S_R$ *is a disjoint union* $S^+ \cup S^-$ *where* S^+ *and* S^- *are path-connected open subsets of* S *mapped isometrically onto each other by* R.

Proof. The only fact we need about R is that at each fixed point there is a small neighbourhood homeomorphic to the unit disc in the plane and the reflection is given by $(x, y) \to (x, -y)$ in this disc. In particular, the x-axis is fixed and the upper and lower half open discs are interchanged. Pick a distinguished point z_0 in $S - S_R$ and let S^+ be the path component of z_0 in $S - S_R$. Let $S^- = R(S^+)$, we are allowing the possibility that $S^+ = S^-$. Since both of S^+ and S^- are path components they are either equal or disjoint. Also, since $R^2 = id$, then S^+ and S^- are interchanged by R if they are distinct. Now let z be any other point of $S - S_R$ and let $\alpha : [0, 1] \to S$ be a path in S connecting z_0 to z. We may assume that α is smooth and crosses S_R only a finite number of places $z_i = \alpha(t_i)$. Let $z_1 = \alpha(t_1)$ be the first crossing. Then, by looking at the local picture of the path crossing the x-axis transversely, we see that $\alpha(t) \in S^+$ for $0 \leq t < t_1$ and $\alpha(t) \in S^-$ for all sufficiently small $t > t_1$. Since S^- is a path component then $\alpha(t) \in S^-$ for $t_1 < t < t_2$. By the same argument, $\alpha(t) \in S^+$, for $t_2 < t < t_3$. By inductive argument we conclude that $z = \alpha(1)$ lies either in S^+ or S^-. ∎

The rest of this subsection is devoted to discussing the properties of mirrors. The task of determining splitting properties is greatly reduced by Proposition 13. Each edge is equivalent to one of the p, q, or r edges, and so we need only examine the splitting properties of these reflections, according to the Proposition.

Proposition 13 *Suppose that R_1, R_2 are conjugate reflections via $R_2 = gR_1g^{-1}$ for some $g \in G^*$. Then g maps the mirror S_{R_1} of R_1 bijectively onto the mirror S_{R_2} of R_2, and, hence, R_1 is separating if and only if R_2 is.*

This proposition also allows us to find some general conjugacy relations among p, q and r.

Proposition 14 *The following parity conditions guarantee conjugacy relations among p, q and r :*

1. *k is odd implies p and q are conjugate,*
2. *l is odd implies q and r are conjugate,*
3. *m is odd implies r and p are conjugate, and*
4. *$|G|$ is odd implies that p, q and r are conjugate.*

Proof. If $|G|$ is odd then all three of k, l and m are odd. Thus we need only prove the first three statements. By similarity of argument we will only prove the first. Let $k = 2\kappa + 1$. Then

$$\begin{aligned} 1 &= (pq)^\kappa (pq)(pq)^\kappa \\ &= (pq)^\kappa p(qp)^\kappa q \\ &= a^\kappa p a^{-\kappa} q. \end{aligned}$$

Thus $q = a^\kappa p a^{-\kappa}$. ∎

Next we consider some propositions which allows us to construct the ovals and the mirrors. The following Proposition, whose proof is left to the reader, allows us to explicitly identify the edges contained in a mirror.

Proposition 15 *Let $e \in \{p, q, r\}$ be any of basic edges that bound Δ_0, and let $R_e \in \{p, q, r\}$ the corresponding reflection. Let $g \in G^*$ and let R be any reflection in G^*. The the e-type edge $g \cdot e$ of $g\Delta_0$ is in the mirror of R if and only if $gR_eg^{-1} = R$.*

Proposition 16 *Let $R \in G^*$ be a reflection. Then $\mathrm{Cent}_G(R)$ acts simply transitively on the edges of the same type in the mirror of R. If e_p, e_q, and e_r are the numbers of p, q, and r edges in a given oval and f_p, f_q, and f_r the number of ovals containing edges of a given type, respectively, then :*

$$\mathrm{Cent}_G(R) = e_p f_p = e_q f_q = e_r f_r$$

as long as a product is not zero. In particular, the total number of edges of a given type in the mirror of R is either 0 or $|\mathrm{Cent}_G(R)|$.

Proof. For the sake of argument, let us suppose that S_R contains a p-type edge. Let $g, h \in G$ be such that $g \cdot p$ and $h \cdot p$ are two p-type edges in the mirror of R. Then, $hph^{-1} = gpg^{-1} = R$ and $(hg^{-1}) \cdot (g \cdot p) = h \cdot p$. But, $hg^{-1}Rgh^{-1} = hph^{-1} = R$ so $hg^{-1} \in \mathrm{Cent}_G(R)$. Thus any p-type edge in S_R can be moved to any other p-type edge in S_R by an element of $\mathrm{Cent}_G(R)$. Now suppose that e is any p-type edge in S_R. This can happen if and only $R \cdot e = e$, or $R_e = R$. If $g \in \mathrm{Cent}_G(R)$, then, $R_{g \cdot e} = gR_eg^{-1} = gRg^{-1} = R$. Thus $g \cdot e$ must also lie in the S_R. It follows then that $\mathrm{Cent}_G(R)$ permutes the p-type edges of S_R transitively. Since G permutes the p-type edges of S simply transitively, then, $\mathrm{Cent}_G(R)$ also permutes the p-type edges of S_R simply transitively. Now by what we have just proven, $\mathrm{Cent}_G(R)$ permutes transitively all the ovals of S_R that contain a p-type edge. Therefore it follows from the orbit stabilizer theorem that $\mathrm{Cent}_G(R) = e_p f_p$. The same argument works for the other edges, assuming that S_R contains edges of these types. ■

The mirror of a reflection R can have the following types of ovals that are $\mathrm{Cent}_G(R)$-inequivalent:

1. an oval containing p, q, and r-type edges in equal numbers,
2. an oval containing exactly two types of p, q, and r-type edges in equal numbers,
3. an oval containing exactly one type of p, q, and r-type edges,
4. one oval of type 2 and one of type 3 made up from different edge types,
5. two ovals of type 3 made up of different edge types, or
6. three ovals of type 3 made up of different edge types.

If two edges appear in the same oval then the e and f numbers of Proposition 16 are the same. Furthermore, if two types of edges meet at a vertex of odd order then the ovals for the corresponding mirrors contain both types of edges. This fact and the pattern of edges on an oval can be deduced from the following Proposition.

Proposition 17 *Let $\mathcal{O}$ be an oval and $g \cdot e$ an edge in $\mathcal{O}$ and $g \cdot v$ an endpoint of $g \cdot e$, where $g \in G$, $e \in \{p, q, r\}$, and $v \in \{P, Q, R\}$. Let $h \in \{gag^{-1}, gbg^{-1}, gbg^{-1}\}$ be the generator of $G_v = \{y \in G : yv = v\}$. Let $e' \in \{p, q, r\}$ be the other edge that meets e at v in the master tile. Let t be the order of h and let $t = 2\tau$ or $s = 2\tau + 1$, depending on the parity of h. Then the oval $\mathcal{O}$ has the two following edges at gv depending on parity of the vertex v.*

1. *If v is an even vertex, then the two edges of $\mathcal{O}$ meeting at $g \cdot v$ are $g \cdot e$ and $(gh^{\tau}) \cdot e$.*
2. *If v is an even vertex, then the two edges of $\mathcal{O}$ meeting at $g \cdot v$ are $g \cdot e$ and $(gh^{-\tau}) \cdot e'$, if e' follows e in the clockwise ordering of the edges of the master tile. Otherwise they are $g \cdot e$ and $(gh^{\tau}) \cdot e'$.*

More complete discussions of patterns of edges on ovals, from which a proof of the above may be derived, are given in [5] and [9]. Examples of using the proposition for constructing mirrors will be given in the next section.

The Reflective Walk Algorithm (RWA) There is a group-theoretic algorithm that may be used to determine whether a given tiling splits or not at a given mirror. Given a master tile and a mirror along which the surface is being cut we look to see how many tiles, after splitting, are in the same component of the surface as the master tile. If the total count is less than $|G^*|$ then the surface splits at the mirror. The following proposition, leads to the Reflective Walk Algorithm, developed by Belk [2], which demonstrates this principle.

Proposition 18 *Let Δ_0 be a tile on the surface and let $g \in G^*$ such that $\Delta_j = g\Delta_0$. Then Δ_0 and Δ_j are in the same component of S after splitting along the mirror S_R of R if and only if there are $d_1, \ldots, d_i \in \{p, q, r\}$ such that $g = d_1 \cdots d_i$ and $d_1 \cdots d_i \neq Rd_1 \ldots d_{i-1}$ for any $1 \leq i \leq j$.*

We repeat the nub of the proof since it is so important for the rest of the paper. Let $\Delta_i = d_1 \cdots d_i \Delta_0$. Then $(d_1 \cdots d_{i-1})d_i(d_1 \cdots d_{i-1})^{-1}$ is a reflection in the edge $(d_1 \cdots d_{i-1}) \cdot d_i$ of $\Delta_{i-1} = d_1 \ldots d_{i-1}\Delta_0$ of type d_i. The reflection in this edge is $(d_1 \cdots d_{i-1})d_i(d_1 \cdots d_{i-1})^{-1}$ and so we cross over edge $(d_1 \ldots d_{i-1}) \cdot d_i$ of tile $d_1 \ldots d_{i-1}\Delta_0$ by the reflection $(d_1 \cdots d_{i-1})d_i(d_1 \cdots d_{i-1})^{-1}$ to get to the next tile, viz.,

$$\begin{aligned}\Delta_i &= d_1 \cdots d_{i-1} d_i \Delta_0 \\ &= (d_1 \cdots d_{i-1})d_i(d_1 \cdots d_{i-1})^{-1} d_1 \cdots d_{i-1}\Delta_0 \\ &= (d_1 \cdots d_{i-1})d_i(d_1 \cdots d_{i-1})^{-1}\Delta_{i-1}.\end{aligned}$$

Now tiles Δ_{i-1} and Δ_i are separated by the mirror S_R of R if and only if the reflection $(d_1 \ldots d_i)d_i(d_1 \ldots d_{i-1})^{-1}$ in the common edge $(d_1 \ldots d_{i-1}) \cdot d_i$ equals R. But $(d_1 \cdots d_{i-1})d_i(d_1 \cdots d_{i-1})^{-1} = R$ if and only if $d_1 \cdots d_i = Rd_1 \cdots d_{i-1}$.

Remark 19 *The sequence of triangles $\Delta_0, \Delta_1, \ldots, \Delta_j$ is called an (j-step) reflective walk on the surface since for each i, $0 < i < j$, $\Delta_{i-1} \cap \Delta_i$ is the edge $(d_1 \ldots d_{i-1}) \cdot d_i$ and one transits from Δ_{i-1} to Δ_i by crossing the common edge $\Delta \cap \Delta_i$ by means of the reflection $(d_1 \ldots d_i) d_i (d_1 \ldots d_{i-1})^{-1}$.*

We denote by S_R^+ the set of tiles in the same component of $S - S_R$ as Δ_0. Abusing notation, we also denote by S_R^+ the elements in G^* labelling the tiles in the same component $S - S_R$ as Δ_0. Here is a result relating the size of S_R^+ to the separability of the surface.

Proposition 20 *Let G^* be the tiling group of a surface S, and cut S along the mirror S_R of the reflection R. Then S does not split at R if and only if $\left|S_R^+\right| > |G|$.*

Remark 21 *Now $\left|S_R^+\right| = |G|$ or $\left|S_R^+\right| = |G^*|$, by Proposition 12. The above proposition is stated in such a way that we conclude that R is non-separating as soon as we count more than $|G|$ elements in S_R^+.*

Propositions 18 and 20 example gives rise to an algorithm, which we shall call the "Reflective Walk Algorithm" (RWA) for determining whether a reflection R splits a surface. The algorithm only depends on calculation in the group G^*. We first describe the version of the algorithm developed by Belk [2].

Reflective Walk Algorithm (RWA)

1. Set $j = -1, V_{-1} = \phi$, $V_0 = \{1\}$.
2. Increment j to $j + 1$, set $W_j = V_j - V_{j-1}, W = \phi$.
3. Complete the following:
    (a) for $g \in W_j$, if $gp \neq Rg$ then $W = W \cup \{gp\}$,
    (b) for $g \in W_j$, if $gq \neq Rg$ then $W = W \cup \{gq\}$,
    (c) for $g \in W_j$, if $gr \neq Rg$ then $W = W \cup \{gr\}$.
4. Set $V_{j+1} = V_j \cup W$.
5. If $|V_{j+1}| > |G|$ then quit, concluding that S does not split at S_R.
6. If $|V_{j+1}| = |V_j|$ then quit, concluding that S does split at S_R.
7. Go back to 2.

The algorithm is easily implemented in Magma, see the script IsSplit.mgm by Belk at the website [11]. The set of all tiles that can be reached in exactly j steps of a reflective walk, not crossing the mirror of R is $\{g\Delta_0 : g \in W_j\}$ and V_j corresponds to the analogous set of tiles reached in j steps or fewer. Step 3 is the crucial part where Proposition 18 is used. Step 6 can only occur if no new tiles can be added. In this case we have completely counted all tiles in the component of Δ_0. If the component is the entire surface we will have already exited at 5, since $|G| > 1$.

The algorithm may be improved by taking a shortcut, guaranteed by the following proposition.

Proposition 22 *If $R \in S_R^+$ then R does not split at R.*

Proof. Let S^+ and S^- be as defined in the proof of Proposition 12. We may assume that S^+ is chosen to contain the interior Δ_0° of Δ_0. By definition $R\Delta_0^\circ \subseteq S^-$. But if $R \in S_R^+$ then $R\Delta_0^\circ \subseteq S^+$ and so $S^+ = S^-$. ■

To incorporate the shortcut, we should modify the above algorithm by replacing Step 5 with the following Step $5'$

5' If $|V_{j+1}| > |G|$ or $R \in W$ then quit, concluding that S does not split at S_R.

A modification of splitting script IsSplit.mgm is IsSplitS.mgm, at the same website. Since most surfaces do not split then there often a potential to "short circuit" and end quickly. Is is not generally known how much faster this algorithm is since the original script worked well for most examples examined. Moreover, our proofs follow the original algorithm.

Fixed point tests An alternative method for showing that a surface does not split at a reflection is to use fixed points of involutions. We may use Proposition 23 following as a test to prove that a reflection is not separating. Though it is computationally easier than algorithm the RWA, it is useless in trying to establish that a reflection is separating. The technique was used in [6] where proofs of the basic facts may be found. Examples of using the proposition are given at the end of the next section.

Proposition 23 *Let R be a separating reflection in G^* and let $h \in G$ be an involution fixing a vertex of even order on the mirror of R (h must the unique element of order 2 in the G-stabilizer of the vertex). Then, if the mirror of R has t ovals, then h fixes at most $2t$ points on S.*

We have already discussed mirrors in the last subsection, Now let us identify the number of fixed points for an element $g \in G$ when G is abelian. A non-abelian formula is also easily worked out and given in [4].

Proposition 24 *Let $g \in G$, where G is an abelian conformal tiling group for a (k, l, m)-generating triple (a, b, c). Define*

$$\delta_a(g) = 1, \text{ if } g \in \langle a \rangle\,,\ \delta_a(g) = 0, \text{ otherwise,}$$
$$\delta_b(g) = 1, \text{ if } g \in \langle b \rangle\,,\ \delta_b(g) = 0, \text{ otherwise,}$$
$$\delta_c(g) = 1, \text{ if } g \in \langle c \rangle\,,\ \delta_c(g) = 0, \text{ otherwise.}$$

Then the number of fixed point of g is given by:

$$|S_g| = |\{x \in S : gx = x\}| = \frac{|G|}{k}\delta_a(g) + \frac{|G|}{l}\delta_b(g) + \frac{|G|}{m}\delta_c(g).$$

Proof. Suppose that hR is an R-type point then $ghR = hR$ implies that $h^{-1}ghR = R$ so that $h^{-1}gh \in \langle a \rangle$ or $g \in \langle a \rangle$. Thus either g fixes all $\frac{|G|}{k}$ of the R-type points or none of them. The formula now follows from similar remarks applied to P-type and Q-type points and the fact that g can fix no non-vertex points of S. ■

4 Illustrative examples

"Big wheel" examples To make the foregoing more concrete, let us examine in detail the $(4, 8, 8)$-tilings by $G = \mathbb{Z}_8$ of a genus 3 surface, including construction of the surface from a fundamental region. Since $G = \mathbb{Z}_8$, $G^* \backsimeq D_8 = \langle x, y | x^2 = y^8 = 1, y^x = y^{-1} \rangle$ with x representing the action of θ and y corresponding to the generator 1 of $\mathbb{Z}_8$. For $(4, 8, 8)$-tilings by this group there are two automorphism classes of generating triples: $[6, 1, 1]$ and $[2, 5, 1]$. The values of $a, b, c, p, q,$ and r are determined by the generating triple $[a, b, c]$ as follows.

$[a, b, c]$	a	b	c	p	q	r
$[6, 1, 1]$	y^6	y	y	y^6x	x	$y^{-1}x$
$[2, 5, 1]$	y^2	y^5	y	y^2x	x	$y^{-5}x$

For geometric reasons, it turns that it is more convenient to represent the elements of G^* by c^s and $c^s r$ than in terms of x and y so we rewrite the table as:

$[a, b, c]$	a	b	c	p	q	r
$[6, 1, 1]$	c^6	c	c	$c^{-1}r$	cr	r
$[2, 5, 1]$	c^2	c^5	c	$c^{-1}r$	c^5r	r

Since G is cyclic we may take as a fundamental region of the surface the wheel of 16 triangles in Figure 6. We choose the master tile Δ_0 to be the triangle in the first quadrant lying on the x-axis. We let the right circular edge of this tile be q, the top edge r, and the bottom edge on the axis be p. The origin will then be the point Q, R will be the other point on the x-axis, and P the third point in the interior of the quadrant. With this labelling the master tile has the same orientation as the standard example previously discussed. The G^* action is given by: c is a counter-clockwise rotation about the origin P through $\pi/8$ radians and r the reflection in the top edge of the master tile. The surface is constructed by identifying pairs of the q-type edges around the rim of the wheel as discussed below. Different generating triples will yield different identification schemes and hence potentially geometrically distinct surfaces.

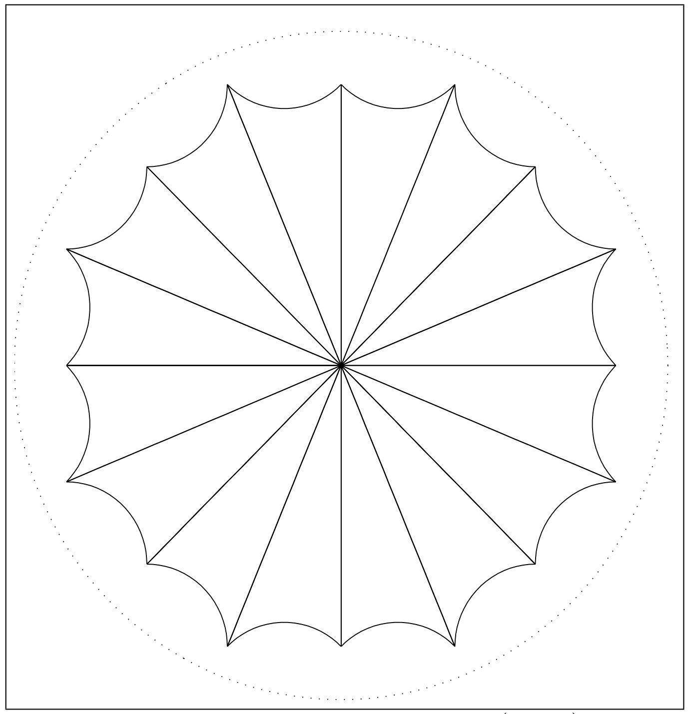

Figure 6. Fundamental region - genus 3, $(4, 8, 8)$-tiling

We now number the tiles starting at 0 with the master tile and counting counter-clockwise. Then we have

$$\Delta_{2s} = c^s \Delta_0,$$
$$\Delta_{2s+1} = c^s \Delta_1 = c^s r \Delta_0.$$

The surface is constructed by identifying pairs of the q-type edges around the rim of the wheel. The identifications will give us pairs of tiles interchanged by reflections in the q-type edges. According to formula 3 and the equations above, the q-type reflections in the various triangles are given by the following table

$[a, b, c]$	Δ_{2s}	Δ_{2s+1}
$[6, 1, 1]$	$c^s q c^{-s} = c^{2s} q = c^{2s+1} r$	$c^s r q r c^{-s} = c^s c^{-1} c^s r = c^{2s-1} r$
$[2, 5, 1]$	$c^s q c^{-s} = c^{2s} q = c^{2s+5} r$	$c^s r q r c^{-s} = c^s c^{-5} c^s r = c^{2s-5} r$

Thus, for example, the $[6, 1, 1]$-action yields the following reflection pairing in the q-type edge of Δ_{2s}

$$c^{2s+1} r \Delta_{2s} = c^{2s+1} r c^s \Delta_0 = c^{s+1} r \Delta_0 = \Delta_{(2s+3) \bmod 16}.$$

Here are all the interchanges for the two different actions

$[a,b,c]$	Δ_{2s}	Δ_{2s+1}
$[6,1,1]$	$\Delta_{2s} \leftrightarrow \Delta_{(2s+3) \bmod 16}$	$\Delta_{2s+1} \leftrightarrow \Delta_{(2s-2) \bmod 16}$
$[2,5,1]$	$\Delta_{2s} \leftrightarrow \Delta_{(2s+11) \bmod 16}$	$\Delta_{2s+1} \leftrightarrow \Delta_{(2s-10) \bmod 16}$

or in table form.

q

$[6,1,1]$	$[2,5,1]$
$\Delta_0 \leftrightarrow \Delta_3$	$\Delta_0 \leftrightarrow \Delta_{11}$
$\Delta_2 \leftrightarrow \Delta_5$	$\Delta_2 \leftrightarrow \Delta_{13}$
$\Delta_4 \leftrightarrow \Delta_7$	$\Delta_4 \leftrightarrow \Delta_{15}$
$\Delta_6 \leftrightarrow \Delta_9$	$\Delta_6 \leftrightarrow \Delta_1$
$\Delta_8 \leftrightarrow \Delta_{11}$	$\Delta_8 \leftrightarrow \Delta_3$
$\Delta_{10} \leftrightarrow \Delta_{13}$	$\Delta_{10} \leftrightarrow \Delta_5$
$\Delta_{12} \leftrightarrow \Delta_{15}$	$\Delta_{12} \leftrightarrow \Delta_7$
$\Delta_{14} \leftrightarrow \Delta_1$	$\Delta_{14} \leftrightarrow \Delta_9$

Since we will also need them, let us also record the interchanges for p and r, which are quite simple.

p

$[6,1,1]$	$[2,5,1]$
$\Delta_0 \leftrightarrow \Delta_{15}$	$\Delta_0 \leftrightarrow \Delta_{15}$
$\Delta_2 \leftrightarrow \Delta_1$	$\Delta_2 \leftrightarrow \Delta_1$
$\Delta_4 \leftrightarrow \Delta_3$	$\Delta_4 \leftrightarrow \Delta_3$
$\Delta_6 \leftrightarrow \Delta_5$	$\Delta_6 \leftrightarrow \Delta_5$
$\Delta_8 \leftrightarrow \Delta_7$	$\Delta_8 \leftrightarrow \Delta_7$
$\Delta_{10} \leftrightarrow \Delta_9$	$\Delta_{10} \leftrightarrow \Delta_9$
$\Delta_{12} \leftrightarrow \Delta_{11}$	$\Delta_{12} \leftrightarrow \Delta_{11}$
$\Delta_{14} \leftrightarrow \Delta_{13}$	$\Delta_{14} \leftrightarrow \Delta_{13}$

r

$[6,1,1]$	$[2,5,1]$
$\Delta_0 \leftrightarrow \Delta_1$	$\Delta_0 \leftrightarrow \Delta_1$
$\Delta_2 \leftrightarrow \Delta_3$	$\Delta_2 \leftrightarrow \Delta_3$
$\Delta_4 \leftrightarrow \Delta_5$	$\Delta_4 \leftrightarrow \Delta_5$
$\Delta_6 \leftrightarrow \Delta_7$	$\Delta_6 \leftrightarrow \Delta_7$
$\Delta_8 \leftrightarrow \Delta_9$	$\Delta_8 \leftrightarrow \Delta_9$
$\Delta_{10} \leftrightarrow \Delta_{11}$	$\Delta_{10} \leftrightarrow \Delta_{11}$
$\Delta_{12} \leftrightarrow \Delta_{13}$	$\Delta_{12} \leftrightarrow \Delta_{13}$
$\Delta_{14} \leftrightarrow \Delta_{15}$	$\Delta_{14} \leftrightarrow \Delta_{15}$

Now let use the RWA to determine which reflections in the $[6,1,1]$ surface are splitting. Let us first look at q. We need to identify the edges of the mirrors determined by p, q, and r. By Proposition 15 an edge $g \cdot e$, $g \in G$, $e \in \{p,q,r\}$ will lie in the mirror determined by q if and only if

$$gR_eg^{-1} = R_{g \cdot e} = R_q = q.$$

as group elements. Noting that $g \in G$ has the form $g = c^s$, that $c^sR_ec^{-s} = c^{2s}R_e$, then for $e = p, q$, and r the equations become $c^{2s} = qR_e$, or in tabular form:

$[a,b,c]$	$e=p$	$e=q$	$e=r$
$[6,1,1]$	$c^{2s} = qp = a^{-1} = c^2$	$c^{2s} = qq = 1$	$c^{2s} = qr = b = c$
$[2,5,1]$	$c^{2s} = qp = a^{-1} = c^6$	$c^{2s} = qq = 1$	$c^{2s} = qr = b = c^5$

.

Solving the resulting equations for c we see that the mirror of q is $c \cdot p \cup c^5 \cdot p \cup q \cup c^4 \cdot q$. Indeed, all the mirrors are given by:

$[a, b, c]$	$[6, 1, 1]$	$[2, 5, 1]$
S_p	$p \cup c \cdot p \cup c^3 \cdot q \cup c^7 \cdot q$	$p \cup c^4 \cdot p \cup c \cdot q \cup c^5 \cdot q$
S_q	$c \cdot p \cup c^5 \cdot p \cup q \cup c^4 \cdot q$	$c^3 \cdot p \cup c^7 \cdot p \cup q \cup c^4 \cdot q$
S_r	$r \cup c^4 \cdot r$	$r \cup c^4 \cdot r$

Using the RWA we construct and count the triangles that are in the same component as the master tile Δ_0. As in the algorithm description, let U_j be the set of triangles we can get to from Δ_0 in j steps of a reflective walk in which we never cross the mirror of q. Let $V_j = U_0 \cup U_1 \cup \cdots \cup U_j$, the set of tiles we can get to in j steps or less, and let $W_0 = V_0$, $W_j = V_j - V_{j-1}$, $j > 0$, the new tiles at stage j. The tiles in W_j can be reached in j steps but no fewer. The sets V_j will grow in size until they become S_q^+. Now obviously $U_0 = V_0 = W_0 = \{\Delta_0\}$. According to the tables we can get to Δ_1 and Δ_{15} in one step from Δ_0 by $\Delta_1 = r\Delta_0$ and $\Delta_{15} = p\Delta_0$, but not $\Delta_3 = q\Delta_0$, since we must cross the mirror of q to do so. Thus $V_1 = \{\Delta_0, \Delta_1, \Delta_{15}\}$ and $W_2 = \{\Delta_1, \Delta_{15}\}$. At the second stage from Δ_1 we get Δ_2 by crossing $r \cdot p$, Δ_{14} by crossing $r \cdot q$ and Δ_0 by crossing r, except that Δ_2 is disallowed since we must cross $c \cdot p$. From Δ_{15} we potentially get Δ_0, Δ_{12}, and Δ_{14}, except that Δ_{14} is disallowed from this direction since we must cross $c^7 \cdot q$. The new elements obtained are $W_2 = \{\Delta_{12}, \Delta_{14}\}$. The following table shows the growth of V_j.

Growth of $V_j,\ R = q,\ [a, b, c] = [6, 1, 1]$.

step j	W_j = new elements	growth $=\lvert W_j\rvert$	$\lvert V_j\rvert$
0	Δ_0	1	1
1	Δ_1, Δ_{15}	2	3
2	Δ_{12}, Δ_{14}	2	5
3	Δ_{11}, Δ_{13}	2	7
4	Δ_{10}	1	8

So one can see that there are 8 elements in S_q^+. Thus $\left|S_q^+\right| = |G|$, and the tiling splits at q. As a contrast we construct the table for q and $[a, b, c] = [2, 5, 1]$.

Growth of $V_j,\ R = q,\ [a, b, c] = [2, 5, 1]$.

step j	W_j = new elements	growth $=\lvert W_j\rvert$	$\lvert V_j\rvert$
0	Δ_0	1	1
1	Δ_1, Δ_{15}	2	3
2	$\Delta_2, \Delta_4, \Delta_{14}$	3	6
3	$\Delta_3, \Delta_5, \Delta_9, \Delta_{13}$	4	10
4	$\Delta_6, \Delta_8, \Delta_{10}, \Delta_{12}$	4	14
5	Δ_7, Δ_{11}	2	16

Carrying out the analysis four more times we are able to determine all the splitting properties of both surfaces.

$[a,b,c]$	p	q	r
$[6,1,1]$	splits	splits	doesn't split
$[2,5,1]$	doesn't split	doesn't split	doesn't split

"Small wheel" examples Now let us consider two examples where none of a, b or c generate G, and so there is no wheel-like fundamental region. A small interesting example is the $(6, 10, 15)$-tiling of a surface of genus 11, with $G = \mathbb{Z}_{30}$ and $G^* = D_{30} = \langle x, y | x^2 = y^{30} = 1, y^x = y^{-1} \rangle$. Since c has odd order p and q are conjugate and so their mirror will contain edges of both types. Now again to determine which edges lie in which mirrors let $e, e' \in \{p, q, r\}$ be any two edges. An element of G has the form y^s and so $y^s \cdot e$ belongs to the mirror of $R_{e'}$ if and only if $y^s R_e y^{-s} = R_{e'}$ or

$$y^{2s} = R_{e'} R_e$$

Now that amounts to solving the equation $y^{2s} = g$ where g is as given in the following table:

e, e'	p	q	r
p	1	a^{-1}	c^{-1}
q	a	1	b^{-1}
r	c	b	1

(5)

As a and b have even order and c, c^{-1} and y^{2s} has odd order, the mirror of q only contains q-type edges and ovals for the p and r mirrors each contain edges of type p and r. By Proposition 17 the pattern of types in an oval in the mirror of p or r must be *pprr* and *qq* on the q-type mirror. By this we mean that as we traverse an oval the sequence of edge types we see will have repeating pattern *pprr*, or a repeating *qq* pattern for the two different types of mirrors. For any abelian conformal tiling group G, and any reflection $R \in G^*$ the automorphism induced by conjugation by R on G is $g \to g^{-1}$. Thus, in the cyclic case this centralizer has two elements if $|G|$ is even and is trivial if $|G|$ is odd. Therefore, by Proposition 16, it follows that each mirror consists of a single oval and the type pattern is *pprr* (four edges) or *qq* (two edges). We are now ready to apply the fixed point formulas. By construction, both ovals have vertices of even order and $h = y^{15}$ is the unique involution corresponding to these points. Since $h \in \langle a \rangle$, $h \in \langle b \rangle$, but $h \notin \langle c \rangle$ then the number of fixed points of h is given by

$$\begin{aligned}|S_h| &= \frac{|G|}{k}\delta_a(h) + \frac{|G|}{l}\delta_b(h) + \frac{|G|}{m}\delta_c(h) \\ &= \frac{30}{6} \times 1 + \frac{30}{10} \times 1 + \frac{30}{15} \times 0 \\ &= 8.\end{aligned}$$

If any of the reflections were separating, then by Proposition 23 h would have at most two fixed points since the mirrors have only one oval each. This contradiction shows that no reflection is separating.

Now let us double the numbers and consider a $(12, 20, 30)$-tiling of a surface of genus 26, with $G = \mathbb{Z}_{60}$ and $G^* = D_{60} = \left\langle x, y | x^2 = y^{60} = 1, y^x = y^{-1} \right\rangle$. The previous analysis works with he following modifications. The reflections p and r are conjugate because $y^{2s} = c$ has a solution in G, but now the mirrors contain two distinct ovals with pattern types pp, and rr. The reflection q is conjugate to neither p nor r and the mirror of q has a single oval with pattern type qq. Now $h = y^{30}$ belongs to all of $\langle a \rangle$, $\langle b \rangle$, and $\langle c \rangle$, so $|S_h| = \frac{60}{12} \times 1 + \frac{60}{20} \times 1 + \frac{60}{30} \times 1 = 10$. But the number of fixed point could be 2 or possibly 4 if it split at any reflection. Again, we get a similar contradiction and conclusion.

5 Splitting surfaces with cyclic conformal tiling group

Work has been done in [4] and [13] to classify many tiling groups of various genera. There has also been work done in [6] and [2] to check whether a surface with a given tiling splits at a mirror. We are concerned with extending these studies and classifying additional families of tiling groups into splitting and non-splitting categories. Several infinite families of groups appear frequently as orientation preserving groups, so one of these is a logical place to start in our classification. For each genus, a large portion of the groups with “smaller”order are cyclic, that is $G = \mathbb{Z}_n$. This is also an infinite family of groups, each with a simple structure, hence it would be a useful starting point to classify this family. So our goal is now to classify the splitting properties of all cyclic conformal tiling groups. First we note that Harvey has determined exactly which triples can occur for a cyclic group (derived from a more general theorem on all cyclic actions).

Proposition 25 *There is a cyclic action of the group $G = \mathbb{Z}_n$ on a surface S if and only if*

$$n = \mathrm{lcm}(k, l, m) = \mathrm{lcm}(k, l) = \mathrm{lcm}(k, m) = \mathrm{lcm}(l, m)$$

the number of integers in $\{k, l, m\}$ divisible by the highest power of 2 is even.

Before we began looking at the splitting properties we sought to find the full tiling group G^* for $G = \mathbb{Z}_n$. First of all we need an automorphism, θ that takes a to a^{-1} and b to b^{-1}. Since G is the cyclic group, and hence abelian, we know that the map $\theta : g \to g^{-1}$ is an involutory automorphism that takes every element to its inverse (Proposition 2). Thus

$$G^* = D_n = \left\langle x, y | x^2 = y^n = 1, y^x = y^{-1} \right\rangle, \qquad (6)$$

representing G^* as an abstract group. Note that

$$w^2 = 1, \text{ for } w \in G^* - G, \text{ and}, \qquad (7)$$
$$wzw = z^{-1}, \; z \in G, \; w \in G^* - G.$$

We began looking at a simple and common family of tilings by cyclic groups where the (k, l, m)-triangles are isosceles $(k, 2k, 2k)$-triangles, where $n = 2k$ is even. For $n = 4$ there is only one possible $[a, b, c]$ generating triple for the tiling, namely $[2, 1, 1]$. This is the 45-45-90 tiling of the torus. Looking at a picture of this tiling it is obvious that cutting along two of the mirrors will most certainly split the surface. More specifically, this tiling splits at the p and q mirrors but not at r. This is similar to our results in the previous section. Computations using the RWA, implemented in Magma, suggested that the splitting at two mirrors phenomenon of the $(2, 4, 4)$ tiling on the torus generalizes to all $(k, 2k, 2k)$-tilings where the generating triple is of the form $[n - 2, 1, 1]$ or is equivalent to such a triple. (See section 6 for more on the determination of the automorphism classes of the (k, dk, dk)-generating triples, $d \geq 1$.) After many examples were computed we surmised that there were no other cyclic tilings which split at any mirror. Thus we arrive at the following theorem.

Theorem 26 *Let $G = \mathbb{Z}_n$, $n \geq 4$, with a (k, l, m)-generating triple $[a, b, c]$, such that $k \leq l \leq m$ and G is the orientation preserving tiling group for some surface. Then the tiling splits at a mirror if and only if*

$$b = c$$
$$k = \frac{n}{2}$$

so in particular n is even. When this is true, the tiling splits at all p-type and q-type mirrors, but at no r-type mirrors.

We break up the proof into three cases with several subcases each. The first case is where $G = \langle c \rangle$ and $b = c$. In the second case we consider $G = \langle c \rangle$ and $b \neq c$, and in third case none of a, b or c generate G. We prove each of the subcases in a series of claims.

We began by working out the first 3 examples, $\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Z}_8$. Here it is easily shown that $|S_r^+| = |G^*|$ and thus the tiling does not split. In order to show that p and q split in this case we found a function that gave the new elements in S_R^+, where R is p or q. From here we looked at the growth pattern of V_s as it increases in size to S_R^+. We found a function which gives the new elements of the component at each step. By induction we proved that the growth pattern for finding the elements of S_R^+: $1, 2, 2, \ldots, 2, 1$ where there are $k - 1$ 2's. Therefore, $\left|S_R^+\right| = |G|$ and the tiling splits.

In the next case we are to show that where $\langle c \rangle = G$ where $b \neq c$ the tiling does not split at all. We first showed that a specific "half a wheel" consisting of $|G|$ tiles is in S_R^+. We then found a single tile that is not in this half wheel that we can get to from the first half without reflecting over the mirror. Thus $\left|S_R^+\right| > |G|$ and so the tiling does not split.

The last case shows that for any cyclic tiling group where no one element of the generating triple generates G the tiling does not split anywhere. In this case we use a combination of RWA arguments and fixed point arguments.

Remark 27 *In the first two cases we can write all elements of G as c^k and all elements of G^* as c^k or $c^k r$.*

Proof of Case $(k, 2k, 2k)$**,** $G = \langle c \rangle$, $b = c$ Observe that $a = (bc)^{-1} = c^{-2}, a^{-1} = c^2$. Now, in the algorithm the reflections in edges is accomplished by right multiplication by p, q and r. To translate the algorithm into the group elements we need to know how the change the canonical forms of the group elements, when we multiply on the right by p, q, or r, and we need to know when such a transition crosses a p, q or r mirror. This is given in the following table, where the entries in the three test column indicates a forbidden transition across the various mirrors whenever there is a solution to the equation.

g	u	gu	p-test	q-test	r-test
c^s	p	$c^{s-1}r$	$c^{2s} = 1$	$c^{2s} = c^2$	$c^{2s} = c$
c^s	q	$c^{s+1}r$	$c^{2s} = c^{-2}$	$c^{2s} = 1$	$c^{2s} = c^{-1}$
c^s	r	$c^s r$	$c^{2s} = c^{-1}$	$c^{2s} = c$	$c^{2s} = 1$
$c^s r$	p	c^{s+1}	$c^{2s} = c^{-2}$	$c^{2s} = 1$	$c^{2s} = c^{-1}$
$c^s r$	q	c^{s-1}	$c^{2s} = 1$	$c^{2s} = c^2$	$c^{2s} = c$
$c^s r$	r	c^s	$c^{2s} = c^{-1}$	$c^{2s} = c$	$c^{2s} = 1$

(8)

Also, it is useful to know what the mirrors are:

mirror	mirror components
S_p	$p \cup c^k \cdot p \cup c^{-1} \cdot q \cup c^{k-1} \cdot q$
S_q	$c \cdot p \cup c^{k+1} \cdot p \cup q \cup c^k \cdot q$
S_r	$r \cup c^k \cdot r$

(9)

These are easily calculated. For instance $c^s \cdot q$ is an edge of the mirror S_p if and only if

$$\begin{aligned} c^s q c^{-s} &= p, \text{ or} \\ c^{2s} &= pq = a \\ c^{2s} &= c^{-2} \\ c^s &= c^{-1}, c^{k-1}. \end{aligned}$$

The equation to solve $c^{2s} = c^{-2}$ also may be taken directly from p-test column for the transition $c^{2s} \to c^{2s} q$.

Claim 28 *Let* $k = \frac{n}{2}$, $G = \langle c \rangle$, *and assume* $b = c$. *Then the tiling by* G^* *splits at* p.

Proof. First we look at a table similar that in our example.

Growth of $V_j \subseteq S_R^+$ for $R = p$

step j	growth	new elements (W_j)
0	1	c^0
1	2	$c^0 r, c^1 r$
2	2	c^1, c^2
3	2	$c^2 r, c^3 r$
4	2	c^3, c^4
5	2	$c^4 r, c^5 r$
.	.	.
.	.	.

Let $f(j)$ denote the elements added at step j. We claim that

$$f(j) = \begin{cases} \{c^{j-1}r,\ c^j r\} & j \text{ odd} \\ \{c^{j-1},\ c^j\} & j \text{ even} \end{cases} \quad 1 \leq j \leq k-1. \tag{10}$$

In the base case: $j = 1$, we have

$c^0 p = pc^0$ crosses the mirror

$c^0 q = cr$ does not cross the mirror.

$c^0 r = c^0 r$ does not cross the mirror.

So $f(1) = \{c^0 r, c^1 r\}$. Now the inductive step. Suppose the inductive hypothesis is true for $j < k-1$. The elements of $f(j+1)$ are obtained by right multiplying the elements of $f(j)$ by p, q, r such that they do not cross the mirror p and such that they are not elements of any previous $f(j)$. The reference transformations are in 8. In the following table the arrowed elements are the new elements of $f(j+1)$.

case: j is even	case: j is odd
$c^{j-1}p = c^{j-2}r$	$c^{j-1}rp = c^j$
$c^{j-1}q = c^j r$	$c^{j-1}rq = c^{j-2}$
$c^{j-1}r = c^{j-1}r$	$c^{j-1}rr = c^{j-1}$
$c^j p = c^{j-1}r$	$c^j rp = c^{j+1} \leftarrow$
$\rightarrow c^j q = c^{j+1}r$	$c^j rq = c^{j-1}$
$\rightarrow c^j r = c^j r$	$c^j rr = c^j \leftarrow$

(11)

If j is even then $j \geq 2$ and the potential new elements are $c^j r$ and $c^{j+1}r$. Now the transition $c^j \rightarrow c^j r$ is always allowed, since the test equation $c^{2j} = c^{-1}$ has no solutions. The transition $c^j \rightarrow c^j q = c^{j+1}r$ will be prohibited when the test equation $c^{2j} = c^{-2}$ has solutions, namely $j = -1, k-1$. Since $1 \leq j+1 \leq k-1$ then these restrictions do not apply. For odd j the transitions given by the arrows are always allowed for j in the given range The only time when $c^{j-1}r \rightarrow c^{j-1}rq = c^{j-2}$ could yield a new element is when $j = 1$, but then this is disallowed. Thus it follows that

$$f(j+1) = \begin{cases} (c^j r,\ c^{j+1}r) & j+1 \text{ odd} \\ (c^j,\ c^{j+1}) & j+1 \text{ even} \end{cases}$$

and the claim holds for all $1 \leq j \leq k-1$. When $j = k-1$, the element $c^j q = c^{j+1}r$ corresponds to a forbidden transition and $f(k) = \{c^{k-1}r\}$ for odd $k-1$. In turn $f(k+1) = \{\}$. Similarly, if $k-1$ is even then $c^j rp = c^k$ is disallowed and $f(k) = \{c^{k-1}\}$, $f(k+1) = \{\}$. Thus there are $k+1$ steps with two elements being introduced at each of the $k-1$ middle steps and one new element for each of the other two steps. There are then $2k = |G|$ elements in S_p^+, so the tiling splits at the mirror of p. ∎

Claim 29 *Let $k = \frac{n}{2}$, $G = \langle c \rangle$, and assume $b = c$. Then the tiling by G^* splits at q.*

Proof. By Proposition 13 we may conclude that q is separating if we can find a $g \in G$ such that $p = gqg^{-1}$. We found just such an element in the calculations for the mirror of p. ∎

Claim 30 *Let $k = \frac{n}{2}$, $G = \langle c \rangle$, and assume $b = c$. Then the tiling by G^* does not split at r.*

Proof. The mirror of r consists of the edges r and $c^k \cdot r$. By repeated p-type and r-type reflections we can construct a half-wheel of tiles $\Delta_0, p\Delta_0$, $pr\Delta_0$, $prp\Delta_0$, up until $(rp)^k p\Delta_0$. The tiles have the form $c^{-s}\Delta_0$, $0 \leq s \leq k-1$ and $c^{-s}r\Delta_0 = c^{-s}\Delta_1$, $1 \leq s \leq k$. Now the q-type reflection $c^s qc^{-s}$ moves $c^s\Delta_0$ to

$$c^s qc^{-s}c^s\Delta_0 = c^s q\Delta_0 = c^s qrr\Delta_0 = c^s br\Delta_0 = c^{s+1}r\Delta_0$$

and $c^s rqrc^{-s}$ moves $c^s\Delta_1 = c^s r\Delta_0$ to

$$c^s rqrc^{-s}c^s r\Delta_0 = c^s rq\Delta_0 = c^s b^{-1}\Delta_0 = c^{s-1}\Delta_0.$$

Since the mirror of r contains no q reflections we may use $s = 0$ to transport Δ_0 to $c\Delta_1$ which is not in our half wheel. Thus $|S_r^+| > |G|$ and the reflection is not separating. ■

Proof of Case $G = \langle c \rangle$, $|G|$ is even In this case we are assuming that $n = 2u$ is even and either that (k, l, m) is not $(k, 2k, 2k)$, or that $(k, l, m) = (k, 2k, 2k)$ and $b \neq c$. Let $b = c^t$ for some $t \in \mathbb{Z}$, $1 \leq t < n$, so that $a = (bc)^{-1} = c^{-t-1}$, and $a^{-1} = c^{t+1}$. Exactly of one of t and $t+1$ is odd and we may assume it is t. For, the (l, k, m)-triple $[b, a, c]$ will satisfy the condition, and we may translate the results of one triple to the other by relabelling. Indeed we will show that no mirrors split, in our assumed case applying the no mirrors split in the other case. We determine the transition rules and the mirrors as we did previously:

g	u	gu	p-test	q-test	r-test
c^s	p	$c^{s-1}r$	$c^{2s} = 1$	$c^{2s} = c^{t+1}$	$c^{2s} = c$
c^s	q	$c^{s+t}r$	$c^{2s} = c^{-(t+1)}$	$c^{2s} = 1$	$c^{2s} = c^{-t}$
c^s	r	$c^s r$	$c^{2s} = c^{-1}$	$c^{2s} = c^t$	$c^{2s} = 1$
$c^s r$	p	c^{s+1}	$c^{2s} = c^{-2}$	$c^{2s} = c^{t-1}$	$c^{2s} = c^{-1}$
$c^s r$	q	c^{s-t}	$c^{2s} = c^{t-1}$	$c^{2s} = c^{2t}$	$c^{2s} = c^t$
$c^s r$	r	c^s	$c^{2s} = c^{-1}$	$c^{2s} = c^t$	$c^{2s} = 1$

(12)

mirror	mirror components
S_p	$p \cup c^u \cdot p \cup c^{-(t+1)/2} \cdot q \cup c^{u-(t+1)/2} \cdot q$
S_q	$c^{(t+1)/2} \cdot p \cup c^{u+(t+1)/2} \cdot p \cup q \cup c^u \cdot q$
S_r	$r \cup c^u \cdot r$

(13)

Claim 31 *Suppose that $G = \langle c \rangle$, $n = 2u$, and if $(k, l, m) = (k, 2k, 2k)$ then $b \neq c$. Then neither p nor q is separating.*

Proof. As in the proof Claim 29 we need only prove this for p. Similar to the proof of Claim 30 the component of Δ_0 contains the "upper" half wheel consisting of the even tiles $c^s\Delta_0$, $0 \leq s \leq u-1$ and odd tiles $c^s\Delta_1 = c^s r\Delta_0$,

$0 \le s \le u-1$. Now consider the action of q-type reflections on tiles. According to the table 12 $c^s q c^{-s}$ moves $c^s\Delta_0$ to $c^{s+t} r\Delta_0 = c^{s+t}\Delta_1$, or by calculation,

$$c^s\Delta_0 \to c^s q c^{-s} c^s \Delta_0 = c^s q\Delta_0 = c^s br\Delta_0 = c^{s+t} r\Delta_0 = c^{s+t}\Delta_1. \tag{14a}$$

Similarly $c^s\Delta_1 = c^s r\Delta_0$ is moved as follows

$$c^s\Delta_1 = c^s r\Delta_0 \to c^{s-t}\Delta_0. \tag{15}$$

We need only pick an allowed value of s,

$$0 \le s \le u-1, \tag{16}$$

that so that reflection in the q-edge of $c^s\Delta_0$ or $c^s\Delta_1$, moves a tile from the upper half wheel to the lower half wheel. If we move an even upper tile to an odd lower tile we need

$$u \le s+t \le 2u-1 \bmod 2u, \tag{17}$$

and if we move an odd upper tile to an even lower tile we need

$$-u \le s-t \le -1 \bmod 2u \tag{18}$$

Moreover, according to the table 12, for an even tile we cannot have $c^{2s} = c^{-(t+1)}$, and for an odd tile, we cannot have $c^{2s} = c^{t-1}$. Thus the forbidden values of s in the range $0 \le s \le u-1$ are

$$s \ne u - \frac{t+1}{2} \tag{19}$$

for even tiles and

$$s \ne \frac{t-1}{2} \tag{20}$$

for odd tiles. Now t satisfies $1 \le t \le 2u-2$ since $b \ne 1, c^{-1}$, Moreover, if $t=1$ then $a^{-1} = c^2$ and $o(a) = n/2$ since n is even. However, we have explicitly assumed that we do not have this case. Thus $2 \le t \le 2u-2$, and indeed since t is odd,

$$3 \le t \le 2u-3. \tag{21}$$

Now if $t \ge u$ and $s=0$ then condition 17 is satisfied and also condition 19 is satisfied since t satisfies 21. Now suppose that $t \le u$ and $s=0$ then condition 18 is satisfied and also condition 20 is satisfied since t satisfies 21. Thus in every case we are able to cross from the upper half wheel to the lower half wheel. ■

Claim 32 *Suppose that $G = \langle c \rangle$, $n = 2u$. Then, r is not separating.*

Proof. Again, following the proof of Claim 30 the component of Δ_0 contains the half wheel of tiles $\{c^{-s}\Delta_0, 0 \le s \le u-1\} \cup \{c^{-s}\Delta_1, 1 \le s \le u\}$. The q-type reflections yield the transitions $c^s\Delta_0 \to c^{s+t}\Delta_1$ and $c^s\Delta_1 \to c^{s-t}\Delta_0$, and there are no forbidden q reflections to worry about. If $t < u$ then pick $s=0$ to carry the tile $\Delta_0 \to c^t\Delta_1$. If $t \ge u$ use $s=0$ to carry $\Delta_1 \to c^{-t}\Delta_0 = c^{2n-t}\Delta_0$. But $u \le t \le 2u-2$ implies $2 \le 2n-t \le u$. In both cases we have moved a tile to the other half wheel. ■

Proof of Case $G = \langle c \rangle, |G|$ **is odd** In this case there is a unique solution to the equation $c^{2s} = c^u$ for every $c^u \in G$. Indeed $s = \frac{n+1}{2}u$ since $\frac{n+1}{2}$ is the inverse of 2 mod n. Let v denote this value. Again $b = c^t$ for some $t \in \mathbb{Z}$, $1 \leq t < n$, so that $a = (bc)^{-1} = c^{-t-1}$, and $a^{-1} = c^{t+1}$. The transition table is the same as table 12 and the mirrors are given by:

mirror	mirror components
S_p	$p \cup c^{-v(t+1)} \cdot q \cup c^{-v} \cdot r$
S_q	$c^{v(t+1)} \cdot p \cup q \cup c^{vt} \cdot r$
S_r	$c^v \cdot p \cup c^{-vt} \cdot q \cup r$

(22)

Claim 33 *Let* $G = \langle c \rangle$, *and assume* $|G|$ *is odd. Then the tiling by* G^* *does not split at* $p, q,$ *or* r.

Proof. Since all three of p, q, and r are conjugate, by Proposition 14, we need only show that S does not split at p. Following the proof of Claim 31, our upper half wheel is $\{c^s\Delta_0,\ 0 \leq s \leq v\} \cup \{c^s r\Delta_1,\ 0 \leq s \leq v-1\}$. The bottom part of the upper wheel is bounded by $p \cup c^{-v} \cdot r$. If we transform an even tile from the upper half wheel to the lower half wheel then s needs to satisfy

$$0 \leq s \leq v \tag{23}$$

and also satisfy

$$v \leq s + t \leq n - 1 \bmod n, \tag{24}$$

If we move an odd tile then

$$0 \leq s \leq v - 1 \tag{25}$$

and

$$-v \leq s - t \leq -1 \bmod n \tag{26}$$

The forbidden transitions give us these conditions:

$$s \neq -v(t+1) \tag{27}$$

in transforming from an even to odd tile and

$$s \neq v(t-1) \tag{28}$$

in transforming from odd to even. Now if $t \geq v$ and $s = 0$, then, conditions 23 and 24 are satisfied and condition 27 is also satisfied since $t + 1 \neq 0 \bmod n$. Now suppose that $t \leq v$ and $s = 0$. Then, conditions 25 and 26 are satisfied and condition 28 is satisfied unless $t = 1 \bmod n$. If $t = 1 \bmod n$ then $(k, l, m) = (n, n, n)$ since n is odd. Moreover the forbidden value of s for even tiles and $t = 1$ is $s = -1$ according to 27. Then the transition $c^{v-1}\Delta_0 \to c^v\Delta_1$ with $s = v - 1$ is a permitted transition connecting the upper and lower half wheels. (Of course in the $(k, 2k, 2k)$ case this critical transition was forbidden yielding a separating reflection in that case.) ■

Case $k, l, m < n$ Now, in the remaining case we do not assume that any one of a, b or c generates G, and the handy device of the half wheel is no longer available. We represent all elements of G^* as $a^s b^t$ or $a^s b^t q$, for $0 \leq s \leq k-1$, $0 \leq t \leq l-1$. Here it is more convenient to use q than r, and the representation is not unique. Similar to previous cases, we have the following transition rules and tests:

g	u	gu	p-test	q-test	r-test
$a^s b^t$	p	$a^{s+1} b^t q$	$a^{2s} b^{2t} = 1$	$a^{2s} b^{2t} = a^{-1}$	$a^{2s} b^{2t} = a^{-1} b^{-1}$
$a^s b^t$	q	$a^s b^t q$	$a^{2s} b^{2t} = a$	$a^{2s} b^{2t} = 1$	$a^{2s} b^{2t} = b^{-1}$
$a^s b^t$	r	$a^s b^{t-1} q$	$a^{2s} b^{2t} = ab$	$a^{2s} b^{2t} = b$	$a^{2s} b^{2t} = 1$
$a^s b^t q$	p	$a^{s-1} b^t$	$a^{2s} b^{2t} = a^2$	$a^{2s} b^{2t} = a$	$a^{2s} b^{2t} = ab^{-1}$
$a^s b^t q$	q	$a^s b^t$	$a^{2s} b^{2t} = a$	$a^{2s} b^{2t} = 1$	$a^{2s} b^{2t} = b^{-1}$
$a^s b^t q$	r	$a^s b^{t+1}$	$a^{2s} b^{2t} = ab^{-1}$	$a^{2s} b^{2t} = b^{-1}$	$a^{2s} b^{2t} = b^2$

(29)

Claim 34 *Suppose that $k, l, m < n$ and n is odd. Then S does not split at any of $p, q,$ or r.*

Proof. It suffices to just consider the reflection p and show that $\left|S_p^+\right| > |G|$. First we show, by induction, that S_p^+ contains the b-wheel, i.e., the set of tiles with a corner at the vertex P opposite the side on the master tile. The elements of the b-wheel are all of the form b^t or $b^t q$. Now $q \in S_p^+$ since the test from the table is $1 = a$, yielding a contradiction. Next $b = qr \in S_p^+$ since the test is now $1 = ab^{-1}$ which is also a contradiction. Suppose $b^t \in S_p^+$. Then, $b^t q \in S_p^+$ unless $b^{2t} = a$. But then $G = \langle a, b \rangle = \langle b \rangle$, a contradiction. Next $b^t qr = b^{t+1} \in S_p^+$ unless $b^{2t} = ab^{-1}$, or $b^{2t+1} = a$, another contradiction. By induction $\langle b \rangle \subseteq S_p^+$ and $\langle b \rangle q \subseteq S_p^+$. Now let b^t be any power such that $b^t \notin \langle a \rangle$. We will show by induction that $b^t \langle a \rangle \subseteq S_p^+$. The base step $b^t \in S_p^+$ has already been proven. Now assume that $b^t a^s \in S_p^+$. Then, $b^t a^s p = a^{s+1} b^t q \in S_p^+$ unless $b^{2t} a^{2s} = 1$. Because n is odd, squaring is an automorphism of G and $b^t = a^{-s}$ contradicting $b^t \notin \langle a \rangle$. Next we multiply by q, and $b^t a^{s+1} = a^{s+1} b^t qq$ will be in S_p^+ unless $a^{2s} b^{2t} = a$, we get a similar contradiction. Now let us count. The set of elements of the form $b^t a^s$, with $b^t \notin \langle a \rangle$ is $G - \langle a \rangle$. We also have shown that $\langle b \rangle q \subseteq S_p^+$. Since the sets $G - \langle a \rangle$ and $\langle b \rangle q$ are disjoint then $\left|S_p^+\right| \geq |G - \langle a \rangle| + |\langle b \rangle q| = n - k + l$. Since we have made no assumptions about the ordering of k, l and m let us assume that $l \geq k$. Thus we need only produce one additional element. Consider $b^t a^s p \in S_p^+$. It has already been counted in S_p^+ if $b^t a^s p = b^u q$ for some u. Then $b^{t-u} = a^{1-s}$. But $b^{t-u} \in \langle a \rangle \cap \langle b \rangle$ even though a^{1-s} ranges over $\langle a \rangle$ and $\langle a \rangle$ strictly contains $\langle a \rangle \cap \langle b \rangle$. Thus $\left|S_p^+\right| > |G|$ and S does not split at p. ∎

Claim 35 *Suppose that $k, l, m < n$ and n is even. Then S does not split at any of $p, q,$ or r.*

Proof. Here we will be able to use a fixed point argument similar the fixed point examples in section 4. Again represent G^* as D_n as in the presentation 6. According to Proposition 25 exactly two of k, l, and m are divisible by the highest power of 2 dividing $|G|$, let us assume that they are k and l. Now in the chart 5 $y^{2s} = g$ is not solvable for $g = a, b$ but is solvable for $g = c$. For, a and b have orders divisible by the largest power of 2 dividing n but y^{2s} does not, and the power of 2 dividing c is smaller than the maximum power of 2 dividing n. Therefore p and r are conjugate and q is conjugate to neither. There are now two separate cases, according to the parity of m. The relevant data is presented in the following table. Again $h = y^{n/2}$ denotes the unique involution in G.

parity of m	odd	even
p mirror patterns	$pprr$	pp, rr
q mirror pattern	qq	qq
r mirror patterns	$pprr$	pp, rr
# fixed points of h	$\frac{n}{k} + \frac{n}{l}$	$\frac{n}{k} + \frac{n}{l} + \frac{n}{m}$

By assumption n/k and n/l are odd integers greater than 1 and $n/m \geq 2$. Thus $\frac{n}{k} + \frac{n}{l} \geq 6$ and $\frac{n}{k} + \frac{n}{l} + \frac{n}{m} \geq 8$. If any of the reflection were separating then we could use Proposition 24, noting that each oval has at least one vertex of even order, to conclude that the number of fixed points of h is at most 4, a contradiction. ■

6 Automorphism classes of triples

The $(\frac{n}{2}, n, n)$-triple and its automorphism classes As before we are looking strictly at the tilings where the conformal tiling group is $\mathbb{Z}_n$ and the full tiling group G^* is D_n. As it is possible to have different elements with the same order in $\mathbb{Z}_n$, there can be different generating triples $[a, b, c]$ for the same $(\frac{n}{2}, n, n)$ triple. For example, let us consider $G = Z_{10}$, $G^* = D_{10}$ that generates a $(5, 10, 10)$ tiling on a genus 4 surface. Since $k = 5$, we need a to be an element of order 5 in $\mathbb{Z}_{10}$. So there are four possibilities; $a = 2, 4, 6,$ or $8 \bmod 10$. We also know that b has order 10. There are four possibilities here as well; $b = 1, 3, 7,$ or $9 \bmod 10$. And, since c also has order 10, it has the same possibilities as b. So, there could be $4^3 = 64$ possibilities for our $[a, b, c]$ generating triple. However, this is not the case because we have the additional relation that $abc = 1$ or considering G as an additive group that $a + b + c = 0 \bmod 10$. For example, if $a = 2$ and $b = 3$, then clearly c must be 5, which does not have order 10 in $\mathbb{Z}_{10}$. Thus $[2, 3, 5]$ is not a generating triple for $G = 10$. The following are all generating triples such that $o(a) = 5$, $o(b) = 10$, $o(c) = 10$.

$b \neq c$	$b = c$
$[2, 1, 7], [2, 7, 1]$	$[2, 9, 9]$
$[4, 7, 9], [4, 7, 9]$	$[4, 3, 3]$
$[6, 1, 3], [6, 3, 1]$	$[6, 7, 7]$
$[8, 3, 9], [8, 3, 9]$	$[8, 1, 1]$

However, of all these 12 generating triples, there are only three automorphism classes for this (k, l, m) triple. That is, there are automorphisms that take some of the generating triples to others. For example, if we let $\omega(g) = g^7$ be our automorphism then ω will take $[6,1,3]$ to $[2,7,1]$. Note that there is always an automorphism that takes c to 1. Furthermore, distinct triples with the same value for c are inequivalent. Therefore, we can reduce the previous 12 generating triples to the following 3 inequivalent triples:

$$[2,7,1],\ [6,3,1],\ [8,1,1]. \tag{30}$$

Remark 36 *As we shall see later in this section we get nice clean formulas for the number of automorphism classes of cyclic groups. For isometry classes of tilings the formulas are not as nice but can be easily determined from equation 4 once a set of representatives of automorphism classes of triples is known.*

We quickly noticed that the number of automorphism classes of triples for different $\mathbb{Z}_n$ were different. The first few for the $(\frac{n}{2}, n, n)$ triples are:

$k = n/2$	# *aut. classes*
2	1
3	1
4	2
5	3
6	5
7	4
8	3
.	.

We sought a sequence indexed by k that gives the number of automorphism classes of $(k, 2k, 2k)$ triples for $\mathbb{Z}_{2k}$. We initially conjectured that the sequence would be closely related to Euler's ϕ function, but were unable to make an immediate connection between the two. Sloane's Online Encyclopedia of Integer Sequences [12] gave an exact match for the initial segment of our sequence which we had for the number of automorphism classes in $\mathbb{Z}_{2k}$. The sequence in [12] is defined as follows:

$$S_2(n) = \left| \left\{ \begin{array}{l} \gcd(x,y) \text{ s.t. } 1 \le x \le n \\ \gcd(x,n) = \gcd(y,n) = 1, \\ y - x = 2 \end{array} \right\} \right|$$

After noticing that this sequence seemed to exactly match the number of automorphism classes for $\mathbb{Z}_{2k}$, we conjectured that the number of automorphism classes of (k, dk, dk)-triples for $\mathbb{Z}_{dk}$ will match a similar sequence with a slight tweak, namely $y - x = d$. That is, the number of automorphism classes for $\mathbb{Z}_{dk}$ will match the following sequence:

$$S_d(n) = \left| \left\{ \begin{array}{l} \gcd(x,y) \text{ s.t. } 1 \le x \le n \\ \gcd(x,n) = \gcd(y,n) = 1, \\ y - x = d \end{array} \right\} \right| \tag{31}$$

Though we did not construct a bijection between the pairs in the equations and classes, we did come up with a multiplicative formula for the number of automorphism classes, which we establish in the next section.

Classes of $(\frac{n}{d}, n, n)$-triples If we just trying to determine the number of automorphism classes rather construct them, there are nice multiplicative formulas that allow direct calculation. The following multiplicative functions for $d \geq 1$ count the number of automorphism classes.

$$\Delta_k(p^\alpha) = \begin{cases} p^{\alpha-1}(p-1) & \text{if} \quad p \mid d \\ p^{\alpha-1}(p-2) & \text{if} \quad p \not| d \end{cases}$$

$$\Delta_d(rs) = \Delta_d(r)\Delta_d(s) \text{ if } (r,s) = 1.$$

Example 37 *Let us consider the case when $d = 2$ and $k = 12$. Then we have the $(12, 24, 24)$-triple and the tiling group $G = \mathbb{Z}_{24}$. So we wish to find $\Delta_2(12)$. Now $\Delta_2(12) = \Delta_2(3)\Delta_2(4)$ as* $\gcd(3, 4) = 1$, $\Delta_2(3) = 1$ *and* $\Delta_2(4) = 2$. *So the number of automorphism classes of $(12, 24, 24)$-triples for $\mathbb{Z}_{24}$ is $2 \cdot 1 = 2$. We did find this to be true based on our* MAGMA *calculations which indicated that there were 2 classes of generating triples, represented by* $[10, 13, 1]$ *and* $[22, 1, 1]$.

Theorem 38 *The number of automorphism classes of (d, dk, dk) generating triples for $\mathbb{Z}_{dk}$ is $\Delta_d(k)$. In particular, if d is odd, $\Delta_d(k) =$ 0 for all even k.*

Proof. Let $X_{d,k}$ be the set of generating (k, dk, dk)-generating triples, and let $Y_{d,k} = X_{d,k}/\mathrm{Aut}(\mathbb{Z}_{dk})$. We need to find $|Y_{d,k}|$. Let $[a, b, c]$ be such a generating triple. Since c generates $\mathbb{Z}_{dk}$ then there is a unique $c' \bmod dk$ such that $cc' \equiv 1 \bmod dk$. The vector $c' \cdot [a, b, c] = [c'a, c'b, 1]$ is equivalent to $[a, b, c]$ and has the c component equal to 1. Because c' is unique there is exactly one such representative in each class, so we may now assume that $c = 1$. Since a and b have exact orders k and dk respectively, then $\gcd(a, dk) = d$, $\gcd(b, dk) = 1$. Now $a = dj$ with $\gcd(j, k) = 1$, and $b \equiv -(dj + 1) \equiv d(k - j) - 1 \bmod dk$. Since $\gcd(-(dj+1), kd) = \gcd(dj+1, kd)$, and, the gcd of $dj+1$ and d is automatically 1, then, $\gcd(dj+1, kd) = 1$ if and only if $\gcd(dj+1, k) = 1$. Thus we may identify $Y_{d,k}$ with the set

$$Y_{d,k} \leftrightarrow \{j : 1 \leq j < k, \gcd(j, k) = 1, \gcd(dj + 1, k) = 1,$$

via

$$j \leftrightarrow [dj, d(k - j) - 1, 1].$$

Now consider $k = p^\alpha$. Then $\gcd(j, k) \neq 1 \Leftrightarrow j \equiv 0 \bmod p$. If $p|d$ then $dj+1 \equiv 1 \bmod p$ hence $\gcd(dj + 1, k) = 1$. Thus the number of elements in the above set is $\phi(k) = p^{\alpha-1}(p - 1)$. Now suppose that $p^\alpha \not| k$. If $\gcd(dj + 1, k) > 0$ then $dj+1 \equiv 0 \bmod p$. There is a unique solution h of this equation, considered $\bmod p$, $(0 \leq h < p)$, since d is invertible in the field $\mathbb{Z}_p$. Now consider the $p^{\alpha-1}$ elements $h + tp$, $0 \leq t \leq p^{\alpha-2}$. They all satisfy $1 \leq h + tp < k$, and $\gcd(h + tp, k) = 1$, since $h + ps \equiv h \neq 0 \bmod p$ Therefore, there are $\phi(n) - p^{\alpha-1} = p^{\alpha-1}(p - 2)$ elements j such that $(j, k) = 1$ and $(dj + 1, k) = 1$.

Now suppose $k = rs$, and $\gcd(r, s) = 1$, and suppose we have

$$(i, r) = (id + 1, r) = 1, \quad 1 \leq i \leq r - 1,$$
$$(j, s) = (jd + 1, s) = 1, \quad 1 \leq j \leq s - 1.$$

By the Chinese Remainder Theorem, there are x and y such that

$$x \equiv i \bmod r,$$
$$x \equiv j \bmod s,$$

and

$$y \equiv di + 1 \bmod r,$$
$$y \equiv dj + 1 \bmod s,$$

or

$$y \equiv dx + 1 \bmod r,$$
$$y \equiv dx + 1 \bmod s.$$

Now since $(x, r) = (x, s) = 1$, we have $(x, rs) = 1$. Similarly, $(y, r) = (y, s) = 1$ implies $(y, rs) = 1$. Since r and s are relatively prime and $y \equiv dx + 1 \bmod r$, and $y \equiv dx + 1 \bmod s$, then $y \equiv dx + 1 \bmod rs$. Now we have a map from pairs (i, j) such that $(i, r) = (j, s) = (di + 1, r) = (dj + 1, s) = 1$ to integers x such that $(x, rs) = (dx + 1, rs) = 1$. Obviously, this function is onto, since $x \to (x \bmod r, x \bmod s)$ is the inverse map. Since there exist $\Delta_d(r)\Delta_d(s)$ distinct pairs of elements satisfying the above requirements, we have $\Delta_d(rs) = \Delta_d(r)\Delta_d(s)$. And so we have shown that the multiplicative function $\Delta_d(k)$ yields the number of automorphism classes of (k, dk, dk) triples for $\mathbb{Z}_{dk}$. ∎

7 Partial results for abelian tiling groups

Now that we have classified the splitting properties of all cyclic groups, our next goal is to classify the properties of all abelian groups. This family also appears frequently as a tiling group for every genus and is fairly easy to work with because of its commutativity. From the *Fundamental Theorem of Finitely Generated Abelian Groups* we know that every finite abelian group is isomorphic to a finite direct product of cyclic groups. And because we know that $\langle a, b \rangle = G$ all abelian tiling groups may be represented by a direct product of at most two cyclic groups. Furthermore, we may write each of these abelian tiling groups as $\mathbb{Z}_n \times \mathbb{Z}_{dn}$. This follows from the Fundamental Theorem and that we may simply rearrange the orders and remain isomorphic. Again using Magma we computed a series of examples of abelian tiling groups and tested their splitting properties. The data suggested that the tilings created by these groups only split if their (k, l, m) triple is of the form $(2, 2n, 2n)$ where the conformal tiling group is $\mathbb{Z}_2 \times \mathbb{Z}_{2d}$. However, unlike the cyclic groups it does not seem to matter what the generating triples for the tiling are. The splitting properties are entirely determined by the group G and by the (k, l, m) triple. This seemed true for all other (k, l, m) triples as well. We then set out to prove the following proposition:

Proposition 39 *Let $G = \mathbb{Z}_n \times \mathbb{Z}_{dn}$ be the conformal tiling group for a (k,l,m) tiling. If $k = n = 2$ and $l = m = dn$ with $d > 1$ then the tiling will split at the r mirror but not at p or q mirrors.*

Again we will prove this theorem in a series of Claims.

Claim 40 *The tiling given in Proposition 39 does not split at any p-type mirrors.*

Proof. We shall show that $\left|S_p^+\right| > |G| = 4n$. In order to show this we will show that the b-wheel $\{b^t, b^t q : 0 \leq t < d\} \subseteq S_p^+$, and that S_p^+ contains at least one additional element of the form S_p^+. We proceed by induction. We know that $b^0 \in S_p^+$. So let us now suppose that $b^t \in S_p^+$. Then $b^t q \in S_p^+$ unless $b^t q = pb^t$ or $b^{2t} = a$ a contradiction since $G \neq \langle b \rangle$. Now $b^{t+1} = b^t qr \in S_p^+$ unless $b^t qrqb^{-t} = p$ or $b^{2t} = pqrq = ab^{-1}$ or $b^{2t+1} = a$ again reaching the same contradiction. Therefore $b^t, b^t q \in S_p^+$ for $0 \leq t < 2k$, so $\left|S_p^+\right| \geq 4k$. All other elements of G not in the b-wheel can be written in the form ab^t, for $0 \leq t < 2k$. We need only show that one of them is in S_p^+. Since $d > 1$, $b^2 \neq 1$. Now $b^t p \in S_p^+$ unless $b^t pb^{-t} = p$ or $b^2 = 1$. Next, $ba = bpq \in S_p^+$ unless $bpqpb^{-1} = p$ or $b^2 = pqpq = a^2 = 1$, again reaching the same contradiction. Thus $\left|S_p^+\right| > 4d$ and the tiling does not split at p. ■

Claim 41 *The tiling given in Proposition 39 does not split at any q-type mirrors.*

Proof. In a manner analogous to the proof of Claim 40 one easily shows that the c-wheel is in S_q^+. Then one shows that $ca \in S_q^+$. Thus $\left|S_q^+\right| > 4d$ and the tiling does not split at q. ■

Claim 42 *The tiling given in Proposition 11 splits at all r-type mirrors*

Proof. We need to show $|S_r^+| = |G| = 4d$. In order to show this we will show that S_r^+ contains the half a b-wheel $\{b^t, b^t q : 0 \leq t \leq d-1\}$, the elements $\{ab^t, ab^t q : 0 \leq t \leq d-1\}$ and no others. We proceed by induction. We know that $b^0 \in S_r^+$. Now suppose that $b^t \in S_r^+$. Then $b^t q \in S_r^+$ unless $b^t qb^{-t} = r$ or $b^{2t} = rq = b^{-1}$, or $b^{2t+1} = 1$. But this is impossible since $0 \leq 2t+1 < 2d$. Next since $b^{t+1} = b^t qr \in S_r^+$ unless $b^t qrqb^{-t} = r$ or $b^{2t} = rqrq = b^{-2}$, or $b^{2t+2} = 1$. This also is impossible since $t + 1 \leq d - 1$. Now to show that $ab^t \in S_r^+$ for t in the given range, we multiply $b^t q$ by p. So $ab^t = a^{-1}b^t = b^t qp \in S_r^+$ unless $b^t qpqb^{-t} = r$ or $b^{2t} = rqpq = b^{-1}a$ or $b^{2t+1} = a$ an impossibility. To get $ab^t q$ we multiply by q, and $ab^t q \in S_r^+$ unless $ab^t qb^{-t} a = r$ or $b^{2t} = b^{-1}$ or $b^{2t+1} = 1$. This has been previously ruled out.

Now we need to show that no more new tiles can be added. The twelve cases we need to consider are provided in the following table.

Case	g	u	gs	test
1	b^t	p	ab^tq	
2	b^t	q	b^tq	
3	b^t	r	$b^{t-1}q$	$b^{2t}=1$
4	b^tq	p	ab^t	
5	b^tq	q	b^t	
6	b^tq	r	b^{t+1}	$b^{2t+2}=1$
7	ab^t	p	b^t	
8	ab^t	q	ab^tq	
9	ab^t	r	$ab^{t-1}q$	$b^{2t}=1$
10	ab^tq	p	b^t	
11	ab^tq	q	ab^t	
12	ab^tq	r	ab^{t+1}	$b^{2t+2}=1$

There is an entry in the test column only if there is a possibility of moving to a tile not ready known to be in S_r^+. If the test equation is satisfied then the transition is not permitted. In cases 3 and 9 the value $t=0$ would produce a new tile, but then $b^t=1$. In cases 6 and 12 the troublesome value of t is $d-1$ but $b^{2t+2}=b^{2d}=1$. ■

Other than the $(2,2,2)$-tiling of the sphere by $G=\mathbb{Z}_2\times\mathbb{Z}_2$ and the tilings specified in Proposition 39, we believe that there are no other abelian tilings that split at any mirror. We constructed several examples of these abelian groups and we have yet to find another (k,l,m) tiling where the surface splits at any mirror. We supply a conjecture that these are the only examples in the next section.

8 Further Questions

Our research has lead to questions we could not completely solve during the program and a several others that were related but we did not even start.

1. Classify the splitting behavior of all abelian groups. Indeed prove or disprove the following conjecture:

    - Conjecture: Let G be an abelian tiling group. The the tiling induced by G has separating symmetries only if it is isometrically equivalent to one of the tilings in Proposition 39 or is the $(2,2,2)$ tiling of the sphere.

2. Find the splitting characteristics for other families of groups.

3. Understand splitting better geometrically.

- Conjecture: Let G^* tile a surface σ. Then the tiling splits at p if and only if there is a p-type oval which is a systole (that is, the shortest geodesic on the surface). Similarly for q and r.

4. Extend the theorem on the number of automorphism classes to all cyclic actions.
5. Classify the generating triples of abelian groups up to automorphism.
6. How fast does the short circuited Reflective Walk Algorithm complete.

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