Tilings Which Split at a Mirror

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Abstract

Let ϕ be a reflection on a surface Σ. The *mirror* of ϕ is the fixed point subset $\Sigma_\phi = \{x \in \Sigma : \phi(x) = x\}$, which is a disjoint union of circles . We say that Σ splits at the mirror of ϕ if $\Sigma - \Sigma_\phi$ is disconnected. We further assume that the reflection is a symmetry of a tiling of Σ by triangles. In this paper we investigate a number of conditions on the tiling that guarantee that Σ splits at a mirror.

Contents

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1 Introduction

Let Σ be a compact, orientable surface. Let Θ be a geodesic, kaleidoscopic tiling of Σ by triangles. We'll define this concept in a later section. For now we will just consider examples of such tilings as given by the $(2, 3, 5)$-tiling of the icosahedron shown in Figure 1 and a $(2, 4, 4)$-tiling of the torus shown in Figure 2. Observe that the side of any triangle in the icosahedral tiling is part of a great circle made up of sides of many triangles. Furthermore the reflection in any of these great circles is an isometric mapping of the sphere to itself which takes triangles to triangles. If R is one of these reflections then the fixed point subset $F_R = \{x \in \Sigma : Rx = x\}$ is exactly a great circle, which we call the *mirror* of R. Similar remarks apply to the torus tiling except that, we need put a special metric on the torus so that the reflections are isometries. Alternatively we may work in the universal cover - the Euclidean plane. Furthermore the properties of the mirrors are more complex The "horizontal" and "vertical" mirrors each have two components (think of cutting a bagel in half) but the "inclined mirrors" have only a single component. There is a further difference in these mirrors which constitutes the entire focus of this paper.

Definition 1 *Let R be a reflection of Σ. We say that Σ* splits *at R if $\Sigma \backslash F_R$ has two components. We also say that R is a* separating *reflection.*

For instance, if Σ is a sphere and F_R is an equator, then Σ splits at R into two hemispheres, as illustrated in the case of the icosahedral tiling.. Clearly, every tiling on a surface of genus 0 splits at any mirror. In the torus example the "horizontal" and "vertical" mirrors each have two components and the surface splits along these mirrors. The inclined mirrors have only one components and do not split the surface. Since this is a little difficult to visualize geometrically, we will prove this later. Strangely, splitting becomes very rare when we consider higher genus surfaces. For instance, of the eleven triangle tilings of genus 2 surfaces, only four split at some mirror. Of the nineteen tilings of genus 3 surfaces, only five split at some mirror. Furthermore the mirrors can have a very large number of circles or ovals.

Of course, it would be nice if we had some sort of test to determine whether or not a surface splits at a mirror. Since we are working on surfaces, $\Sigma \backslash F_R$ is connected if and only if it is path connected. Therefore, for every triangle t, we can simply test for a path in Σ from a base triangle t_i to t which doesn't cross F_R. Since the system of vertices, edges and polygons of the tilings is locally finite and exhaust Σ, almost every path in Σ can be associated with a finite path of triangles. (Exceptions include paths which move across a vertex or on an edge and paths which wiggle infinitely around some edge or spiral infinitely towards a vertex. These cases turn out to be inconsequential since they are homotopic to a locally linear path described below.) Therefore, we can determine if $\Sigma \backslash F_R$ is path connected by determining if, for all triangles t, there is a path of triangles from t_i to t which does not cross F_R.

Remark 2 *To get a very clean relationship between paths of triangles and paths on the surface we use the dual graph of the tiling. Let I_t be the incenter of the triangle t. Form the dual graph by connecting I_t and I_u by a line segment if the triangles t and u have a common edge e. This line segment is perpendicularly bisected by e as the local reflection across e is a congruence of triangles. The reflection in e interchanges I_t and I_u. A path of triangles starting and finishing at t_i corresponds to a closed linear path on the dual graph starting and finishing at I_{t_i}. Every path on Σ based at I_{t_i} is homotopic to such a path. For a path on the dual graph the notion crossing a mirror is well defined.*

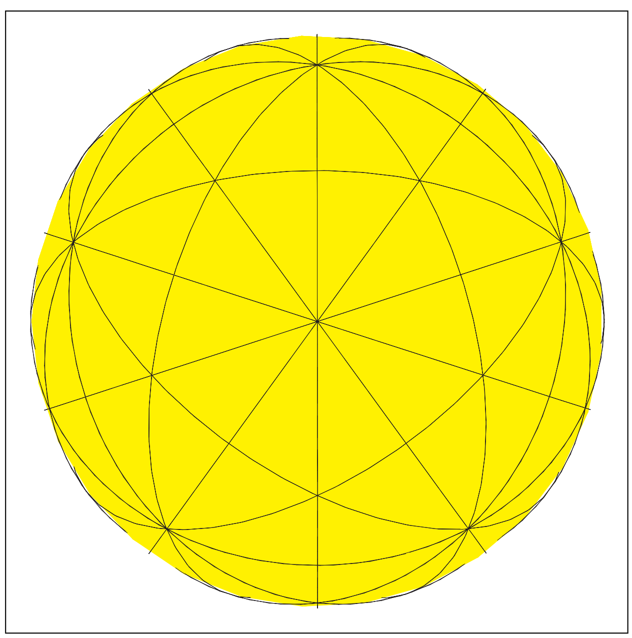

Fig. 1: Icosahedral tiling of the sphere.

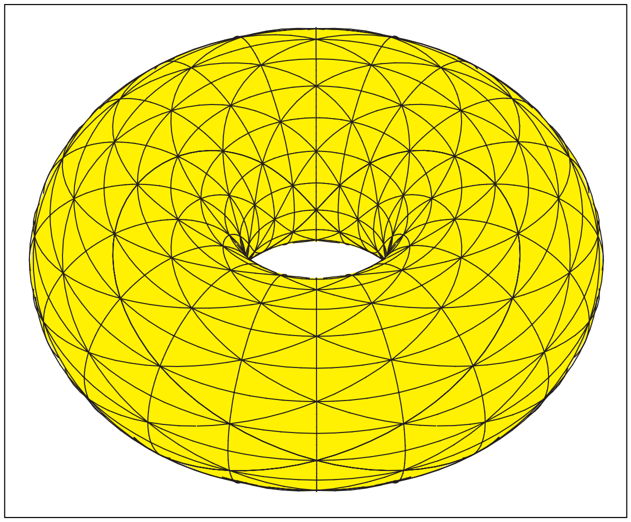

Fig. 2: $(2, 4, 4)$ tiling of the torus.

The tiling group In the examples we assumed that for each edge e there was a reflection R_e of the surface that fixed that edge and interchanged the two triangles whose common boundary is e. All of these reflections generate a group G^* of isometries of the surface whose elements are in 1-1 correspondence to the tiles on the surface. The group G^* and its covering groups acting on the covers of the surface will prove to be extremely useful in analyzing the structure of the tiling, and finding tilings with mirrors that split. The two goals of our study is the following.

> **Goal 1.** Find an efficient group theoretic algorithm to determine whether a given reflection of the tiling is separating or not.
>
> **Goal 2.** Find ways of constructing tilings with mirrors that split, especially by constructing coverings of tilings.

Our principal result for Goal 1 (Theorem 6) is an algorithm, suitable for computer computation, for determining whether a given tiling splits at a mirror. It is not a satisfactory answer since there is no efficient algorithm for large genus surfaces. For Goal 2 a construction is given of a cover $\widetilde{\Sigma} \longrightarrow \Sigma$ such that the mirror of the lift of a reflection splits, or at least is the prime candidate of such a cover. (If we go to the universal cover $U \longrightarrow \Sigma$ the lift of every reflection is separating, but that is not an interesting example.

Associated to the cover $\widetilde{\Sigma} \longrightarrow \Sigma$ there is a covering of groups $\widetilde{G}^* \longrightarrow G^*$. Another group theoretic answer to Goal 1 is $\widetilde{G}^* = G^*$. However the limited time scope of the original project did not allow for development of an algorithm for calculating $\widetilde{G}^*$. See section 8 on further questions.

The paper[3] by Bujulance and Singerman is a good background paper on the separability of reflections. It gives several general methods for determining separability including the graphical Hoare-Singerman theorem given in [4]. Another general group theoretic criterion is given in [2] which used to prove that the $(2,3,7)$-tilings derived from $PSL_2(q)$ Hurwitz action have no splitting mirrors. The discussion from earlier versions of [2] lead to the Theorem 6.

Outline of the report. In section 2 we quickly develop ideas on, tilings the tiling group and discuss the algorithm on detecting a splitting. Sections 3, 4, and 4 introduce the machinery for constructing covers in which mirrors split the covering surface. Sections 6 and 7 discuss the splitting properties of covering groups and covering surfaces introduced in the previous sections. Finally in section 8 we discuss future directions.

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2 Tilings, the tiling group and splitting

Tilings Let Σ be a connected, orientable surface. Suppose that Θ is a collection of open polygons on Σ. Then Θ is a *tiling* of Σ if and only if:

1. The elements of Θ are pairwise disjoint.

2. The surface Σ is the closure of $\cup\Theta$.

Throughout this paper, we shall be considering only *tilings by triangles.* That is, we shall be considering only tilings Θ whose elements are open triangles.

Let t be a triangle in Θ, and consider a single edge e of t. Since e is an edge of a polygon, it is contained in some long, straight curve, known as a *geodesic*. For instance, the geodesics on the plane are the lines, and the geodesics on the sphere are the great circles. For each geodesic g, we assume there is a reflection $R = R_g = R_e$ on the surface such that g is in F_R, the set of fixed points of R. Note that if R exists, it is the unique isometry of Σ fixing e pointwise. On the sphere, the Euclidean plane and the hyperbolic plane, $g = F_R$ for all geodesics g. However, this is not true in general. For example, if g is an outside equator on a torus, then F_R is the union of g and an inside equator g'. In general, F_R, called the *mirror* of the reflection R, may consist of at most $\leq \sigma + 1$ of geodesics or ovals, where σ is the genus of Σ.

When Σ is a surface, the set of reflections R associated with geodesics on the surface generate group under composition, which we denote by G^* and call the *full tiling group*. The $*$ is used to signify that G^* includes reflections. Since we shall want to use G^* to label the triangles, we wish to place conditions on Θ under which G^* acts simply transitively on Θ.

It turns out that two extra conditions are sufficient: the *kaleidoscopic* condition and the *geodesic* condition. The kaleidoscopic condition requires that for each edge there is a reflection R_e, globally defined on Σ, such that for each $\theta \in \Theta$, we have $R(\theta) \in \Theta$. The geodesic condition stipulates that $\Sigma \setminus \cup \Theta$ be a union of mirrors. Specifically, if e is an edge of a triangle in Θ, then F_R is a union of edges of triangles in Θ, where R is the reflection associated with e. These two conditions guarantee that G^* is a group of transformations which acts simply transitively on Θ .

Remark 3 *Note that G^* is* not *the symmetry group of the tiling, since some reflections which preserve the tiling may not be reflections in geodesics in $\Sigma \setminus \cup \Theta$.*

Structure of G^* and G We record some properties about the structure of G^*, derived from the geometry of the tiling. More details on the geometric considerations are given in [1] though our notation differs slightly from that in [1]. Since the angles at a vertex are all equal then the angles of the triangle have measure $\frac{2\pi}{l}$ radians, $\frac{2\pi}{m}$ radians and $\frac{2\pi}{n}$ radians for some integers l, m, n respectively, as shown in Figure 3. (In Figure 3 the triangle is drawn with curved sides since it represents a hyperbolic triangles on a surface of genus greater than 2.) Let p, q, and r also denote the reflections in the sides p, q, and r of the base triangle. Define $a = pq$. It is easily seen to be a counter-clockwise rotation about the vertex R through $\frac{2\pi}{l}$ radians. Similarly, for $b = qr$ and $c = rp$ are counterclockwise rotations about vertex P through $\frac{2\pi}{m}$ radians and about vertex Q through $\frac{2\pi}{n}$ radians, respectively.

From these observations and the fact that reflections have order 2, we get the following:

$$a^l = b^m = c^n = 1, \tag{1}$$

and

$$abc = 1, \tag{2}$$

since $pqqrrp = 1$.

Now $G^* = \langle p, q, r \rangle$ and $G = \langle a, b, c \rangle = \langle a, b \rangle$ is the subgroup consisting of all orientation-preserving elements of G^*. We call G the *OP tiling group*. The subgroup G is normal in G^* of index 2, in fact $G^* = \langle q \rangle \ltimes G$, a semi-direct product. The conjugation action of q on the generators a, b of G induces an automorphism χ satisfying:

$$\chi(a) = qaq = qaq^{-1} = a^{-1}, \tag{3}$$

$$\chi(b) = qbq = qbq^{-1} = b^{-1} \tag{4}$$

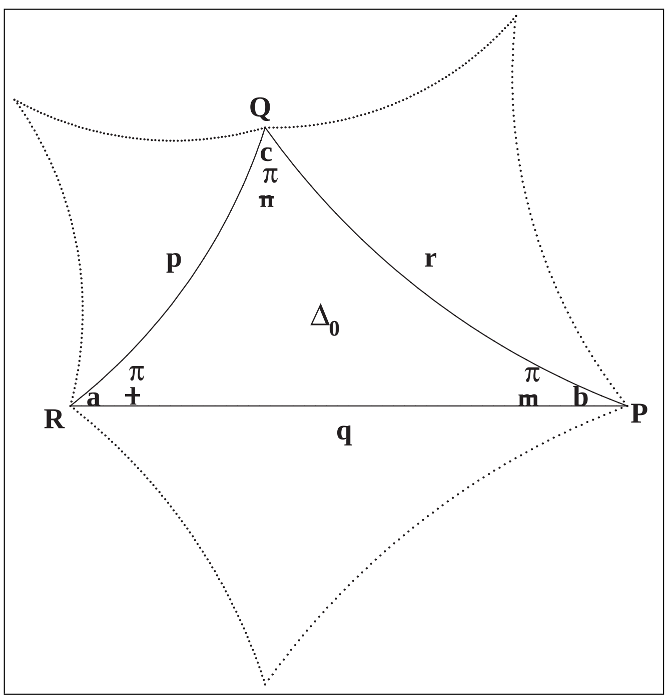

Fig 3: The base triangle and generating reflections.

The relation between the group order G and the genus σ of the surface is given by the Riemann-Hurwitz equation:

$$\frac{2\sigma - 2}{|G|} = 1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right). \tag{5}$$

It follows that the genus is given by:

$$\sigma = 1 + \frac{|G|}{2}\left(1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)\right), \tag{6}$$

and the group order by:

$$|G| = \frac{2\sigma - 2}{1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)}. \tag{7}$$

The triple of elements (a, b, c) of elements from G which generates G and satisfies (1) and (2) is a called a *generating* (l, m, n) *-triple* of G. Just as we may create a triple from a tiling, a tiling may be created from triples as in the following theorem.

Theorem 4 *Let G have a generating (l, m, n)-triple and suppose that the quantity σ defined by (6) is an integer. Suppose in addition, there is an involutary ($\chi^2 = id$) automorphism χ of G satisfying (3) and (4), then the surface S has a tiling T by (l, m, n)-triangles such that tiling group as constructed above is the original G, and such that $G^* \simeq \langle\chi\rangle \ltimes G$.*

Splitting Since the action by G^* is simply transitive, we can label the elements of Θ using elements of G^*. Specifically, let t_i be the chosen base triangle in Θ. If t is any other triangle in Θ, then there is a unique $g \in G^*$ such that $g(t_i) = t$. That is, there exists a bijection $\theta : G^* \longrightarrow \Theta$ such that $\theta(i) = t_i$ and $\theta(g) = g(t_i)$ for all $g \in G^*$.

As above let p, q, and r be the three reflections surrounding $t_{i.}$ as shown in Figure 3. That is p, q, and r are the reflections associated with the three geodesics that border t_i. Then the three geodesics bordering any other triangle $t = g(t_i)$ will be associated with the reflections gpg^{-1}, gqg^{-1}, and grg^{-1}. Thus, if t is any triangle, we can unambiguously talk about the p-type edge, the q-type edge, and the r-type edge of t. Furthermore, since the fixed point subset of every reflection in G^* must contain an edge, every reflection is conjugate to one of p, q, and r. Note that $gpg^{-1}(\theta(g)) = \theta(gpg^{-1}g) = \theta(gp)$, so right-multiplication of g by p amounts to crossing the p-type edge of $\theta(g)$. Similarly, right-multiplication by q or r amounts to crossing the q-type or

r-type edge, respectively. Therefore, a word in p, q, and r, corresponds to a path of triangles on the surface and vice versa. For example, the word $pqpr$ corresponds to the path starting at t_i, then crossing the p-type edge to $\theta(p)$ next crossing the q-type edge to $\theta(pq)$, then crossing the p-type edge to $\theta(pqp)$, and finally crossing the r-type edge to $\theta(pqpr)$. An element of Θ can therefore be thought of as an equivalence class of words in p, q,and r that end up at the same triangle. I.e., G^* is a quotient of the free group generated by p, q, and r and words determine the same triangle if and only if the map to the same element of G^*. The closed paths starting at t_i will correspond to those words that equal the identity in G^*.

Now let us consider splitting at the mirror of F_R in terms of the group G^* and the identification of paths with the words from G^*. Consider a word w whose corresponding path crosses F_R exactly once. If $\Sigma \backslash F_R$ is path connected, there will be another word v which never crosses F_R such that $w \equiv v$ in G^*. In this case, wv^{-1} will be a closed path in G^* which crosses F_R exactly once. Therefore, if Σ does not split at F_R, then there is some closed path of triangles in Θ which crosses F_R an odd number at times. However, the converse is also true, so Σ dos not split at F_R if and only if there is a closed path of triangles in Σ which crosses F_R an odd number of times. Conversely, Σ splits at F_R if and only if every closed path of triangles in Θ crosses F_R an even number of times.

But how can we determine if a path of triangles crosses F_R? Recall that a path of triangles in Σ corresponds to a word in p, q, and r. That is if $\theta(g_0) = t_i, \theta(g_1), \cdots \theta(g_n)$ is a path of triangles, then it corresponds to a word d_n in p, q, and r, where $d_k = g_{k-1}^{-1} g_k$. Then we cross F_R between $\theta(g_{k-1})$ and $\theta(g_k)$ if and only if $g_k = R g_{k-1}$. Therefore:

Proposition 5 *Let $t \in \Theta$, and let $g \in G^*$ so that $t = \theta(g)$. Then t and t_i are in the same component of $\Sigma \backslash F_R$ if and only if there are $d_1 \cdots d_n \in \{p, q, r\}$ such that $g = d_1 \cdots d_n$ in G^* and $d_1 \cdots d_k \neq R d_1 \cdots d_{k-1}$ in G^* for any $1 \leq k \leq n$.*

This approach suggests certain graph-theoretic algorithms to determine if Σ splits at F_R. Given a tiling Θ of a surface, construct a graph whose nodes are triangles in Θ and whose edges are edges in $\Sigma \backslash \cup \Theta$. (This is the dual graph described in Remark 2.) Then remove those edges corresponding to edges in F_R, and use a graph-theoretic algorithm to determine whether the graph is still connected. Furthermore, since the graph is based on a group, we can use certain properties of the graph to speed up the processing. For instance, since G^* is finite, every element of G^* has finite order, so we can construct large cycles in the graph by repeating the same path until we arrive at the identify node.

An algorithm using the dual graph which may be easily translated into a group theoretic algorithm is the following. Let $X_0 = \{t_i\}$ Let X_1 be the set of neighbors of triangles in X_1 which are not obtained by crossing F_R. From the previous discussion these will be $\{pt_i, qt_i, rt_i\}\backslash\{Rt_i\}$. Having defined X_n let us construct the neighbours of the various gt_i not obtained by crossing F_R. For gt_i this will be the set $\{gpt_i, gqt_i, grt_i\}\backslash\{Rgt_i\}$. Thus

$$X_{n+1} = X_n \cup \bigcup_{gt_i \in X_n} \{gpt_i, gqt_i, grt_i\}\backslash\{Rgt_i\}.$$

We will know that we cannot add any more triangles without crossing F_R when $|X_{n+1}| = |X_n|$. At this point $\bigcup_{t \in X_n} t$ is the path component of the base triangle in $\Sigma\backslash F_R$ and F_R will separate if and only if $|X_n| = \frac{|G^*|}{2}$. For practical implementation using Magma we just keep track of the group elements $Y_n = \{g : gt_i \in X_n\}$ and just compute

$$Y_0 = \{i\},\ Y_{n+1} = Y_n \cup \bigcup_{g \in Y_n \backslash Y_{n-1}} \{gp, gq, gr\}\backslash\{Rg\}. \tag{8}$$

Moreover we need only construct $|Y_n|$ until $|Y_n| = |Y_{n-1}|$ or $|Y_n| > \frac{|G^*|}{2}$. We formulate this as a theorem.

Theorem 6 *Let G^* be the full tiling group of a surface Σ, let R be a reflection on Σ and let all other notation be as above. Then Σ, does not splits at R if and only if $|Y_n| > \frac{|G^*|}{2}$ for some n.*

Example 7 *Let us consider the $(2, 4, 4)$ toral examples. Relabel the vertices so that $l = 4$, $m = 2$ and $n = 4$. as suggested by the $45 - 90 - 45$ triangle at the front of the torus. The elements a^2b and bc^2 are a pair of orthogonal translations of the torus of the same order. They must be of the same order since*

$$a(a^2b)a^{-1} = a^{-1}ba^{-1} = bcbbc = bc^2$$

because $a^{-1} = bc$ from 1 The group G the has the structure $\mathbb{Z}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle a \rangle \ltimes (\langle a^2b \rangle \times \langle bc^2 \rangle)$. It follows that there are $|G^*| = 8n^2$ triangles. Pictures of the tiling for $n = 1, 2$ are given in Figures 4 and 5 below. Now let us apply the algorithm to the three cases with $n = 1$ and $R \in \{p, q, r\}$. By manipulating the equations $a^2b = bc^2 = abc = 1$ we get $b = a^2 = b$,

$c = a, p = qa^3$, $r = qa^2$ and hence $G^* = \langle q \rangle \ltimes \langle a \rangle$. We get the following table which shows that p does not split but q and r do split.

	p	q	r
Y_0	$\{id\}$	$\{id\}$	$\{id\}$
Y_1	$\{id, q, qa^2\}$	$\{id, qa^2, qa^3\}$	$\{id, q, qa^3\}$
Y_2	$\{id, q, qa^2, a, a^2, a^3\}$	$\{id, qa^2, qa^3, a\}$	$\{id, q, qa^3, a^3\}$
Y_3	$\{id, q, qa^2, a, a^2, a^3, qa, qa^3\}$	$\{id, qa^2, qa^3, a\}$	$\{id, q, qa^3, a^3\}$

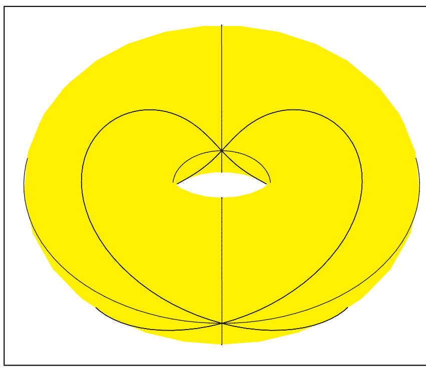

Fig. 4: (2,4,4) tiling - 8 triangles

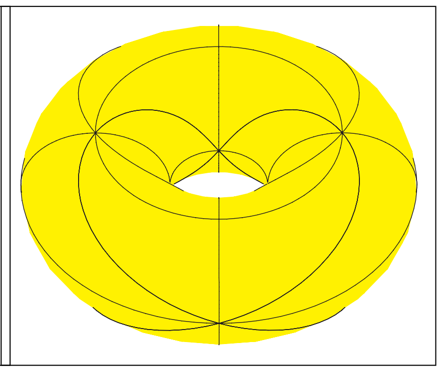

Fig. 5: (2,4,4) tiling 32 triangles

Unfortunately, at this time there is no efficient fully group-theoretic algorithm to determine if a tiling splits. The algorithm above works well for small groups but works more slowly for large groups, and seems to be ineffective for families of groups. For instance, the method used in to determine separability for $PSL_2(q)$-Hurwitz actions in [2] works for an infinite family but has limited applicability. Furthermore, there is as yet no reliable way to produce tilings which split at a mirror, except in special cases such as hyperelliptic curves. Since splitting becomes rarer as the genus of the surface increases, it is not practical to simply produce random tilings and hope that some of them will split. A more systematic way of producing splitting tilings is needed. This topic will occupy the remainder of the paper.

3 The universal cover and fundamental group

Since Σ is a surface, it must have a universal cover U. Furthermore, the geodesic, kaleidoscopic tiling by triangles Θ on Σ must lift to a geodesic, kaleidoscopic tiling by triangles Θ' on U. What is the tiling group Λ^* of this tiling? Like the tiling on Σ, it is generated by the three reflections surrounding the base triangle. As for the relations, a word in p, q, and r is equal to the identity in Λ^* if and only if the path in Σ corresponding to the word is closed and homotopic to the identity path.

Therefore, our strategy for building the relations for Λ^* will consist of defining certain small path deformations which, when combined, can be used to build any homotopy of paths. Suppose that we have some homotopy of paths. Such a homotopy can be built from small transformations, each transformation occurring within some ε-ball. If ε is small enough, then there are only two changes such a small deformation can accomplish: it can move across an edge and back, or it can move around a vertex.

Now we try to determine the group relations resulting from those two operations. If we move across an edge, say a p-type edge, and back, it corresponds to multiplying by p twice on the right. Therefore, we get the relation $p^2 = 1$. Similarly, $q^2 = 1$ and $r^2 = 1$. If we move around a vertex, say a vertex with p-type and q-type edges radiating out of it, it corresponds to right-multiplication by $pqpqpq \cdots pq$. Therefore, we get the relation $(pq)^l = 1$ for some $l \in N$. Similarly, $(qr)^m = (rp)^n = 1$ for some $m, n \in N$. Note that these can also be proven geometrically, see [**?**]. The following is well known though we do not prove it here.

Theorem 8 *Let Λ^* be the tiling group generated by an (l, m, n)- triangle on a simply-connected surface. The Λ^* has the presentation*

$$\Lambda^* = \langle p, q, r | p^2, q^2, r^2, (pq)^l, (qr)^m, (rp)^n \rangle.$$

The subgroup G^* will be a quotient of Λ^*, with the extra relations resulting from closed paths in Σ which are not homotopic to the identity. Specifically, if we let Γ be the set of all $x \in \Lambda^*$, where $\theta(x)$ is a triangle in U whose image in Σ is the base triangle t_i, then $\Gamma \lhd \Lambda^*$ and $G^* = \Lambda^*/\Gamma$. Let $\varphi : \Lambda^* \longrightarrow G^*$ be the resulting homomorphism and let

$$\Gamma \longrightarrow \Lambda^* \xrightarrow{\varphi} G^*$$

be the corresponding exact sequence. Then $\theta(\varphi(x))$ in Σ is the image of $\theta(x)$ under our covering map.

Notation 9 *Sometime we will want to use similar notation (or even the same!) for elements of Λ^* and there images. If the is a strong need to avoid confusion we denote the image $\varphi(x)$ for $x \in \Lambda^*$ by $\overline{x}$.*

But what do universal covers have to do with splitting? The following theorem explains our interest in this subject:

Theorem 10 *Any kaleidoscopic, geodesic tiling by triangles of a simply-connected surface splits at every mirror.*

Proof. Let U be a simply connected surface. Let Θ' be a kaleidoscopic, geodesic tiling of U by triangles, and let $\Lambda^* = \langle p, q, r | p^2, q^2, r^2, (pq)^l, (qr)^m, (rp)^n \rangle$ be its tiling group. We claim that every closed path in Λ^* crosses every mirror an even number of times. Specifically, we claim that each of the six paths specified by the relations crosses every mirror an even number of times.

First, consider the p^2 word. It corresponds to a path starting at the identity and crossing F_p twice. This path crosses every other mirror zero times, so it crosses every mirror an even number of times. The situation with the q^2 and r^2 words is similar.

Now consider the $(pq)^l$ word. When crossing from the $(pq)^k$ to the $(pq)^k p$ triangle, we cross the mirror corresponding to the reflection $(pq)^k p (pq)^{-k} = (pq)^{2k} p$. When we cross from the $(pq)^{k-1} p$ to the $(pq)^k$ triangle, we cross the mirror corresponding to the reflection $(pq)^k p^{-1} (pq)^{1-k} = (pq)^{2k-1} p$. Therefore, the mirrors we cross are $p, (pq)p, \cdots, (pq)^{2l-1} p$. However, $(pq)^l = 1$,so this sequence of mirrors is the same as $p, (pq)p, \ldots, (pq)^{l-1} p, p, (pq)p, \ldots, (pq)^{l-1} p$.

However, $(pq)^l = 1$, so this sequence of mirrors is the same as $p, (pq)p, \cdots, (pq)^{l-1} p, p, (pq)p, \cdots, (pq)^{l-1} p$. Therefore, the $(pq)^l$ path crosses each of the mirrors in $\{p, (pq)p, \cdots, (pq)^{l-1} p\}$ twice, and every other mirror zero times, so it crosses every mirror an even number of times. The situation with the $(qr)^m$ and $(rp)^n$ words is similar. ■

Remark 11 *The above theorem can also be proven by using the classification theorem of the universal covers an the characterization of the reflections.*

Suppose that Σ does not split. Since the tiling of U splits at every mirror, we must somehow lose something in the transition from U to Σ. Specifically, we just have somehow chosen the "wrong" normal subgroup Γ. If we can somehow pin down what went wrong with Γ, we might be able to construct normal subgroups which work, and thus construct tilings that split. In the next section we develop a tool for investigating this.

4 Parity functions and the wreath product

Recall that a surface splits at a mirror F_R if and only if every path of triangles from the identity triangle to itself crosses the mirror an even number of times. Since U splits along every mirror, any closed path of triangles in Θ' must cross every mirror an even number of times.

More formally, if $w = d_1 \cdots d_n$ is a word in p, q,and r, and R is a reflection in Λ^*, we can define the number of times w crosses F_R to be $|\{k|Rd_1 \cdots d_{k-1} = d_1 \cdots d_k\}|$. Let $\mathcal{R}$ be the set of reflections in Λ^*. Then, for every word w, we obtain at a function $\psi_w : \mathcal{R} \to Z_2$ given by $\psi_w(R) = |\{k|Rd_1 \cdots d_{k-1} = d_1 \cdots d_k\}| \bmod 2$. Based on our previous arguments, we would then expect that $\psi_w(R) = 0$ for all the $R \in \mathcal{R}$ whenever $w \equiv 1$ in Λ^*.

Another way to calculate $\psi_w(R)$ is the following. Let $t_i = t_0, t_1, \ldots, t_n$ be the sequence of triangles encountered in the triangle path, i.e., $t_k = d_1 \cdots d_k t_i$. Let $\iota_k = 0 \bmod 2$ if t_k and t_i are in the same component of $U \backslash F_R$ and $\iota_k = 1 \bmod 2$ otherwise. Note that this can be done on any surface as long as Σ splits along the reflection. Now the path crosses F_R as we go from t_k to t_{k+1} if and only if $\iota_{k+1} - \iota_k = 1 \bmod 2$. The total number of times that the path crosses F_R is $\sum_{k=0}^{n-1} \iota_{k+1} - \iota_k = \iota_n - \iota_0$. Thus the $\psi_w(R)$ depends only on the last triangle on not on the path to get there. Now, let v and w be two words in p, q, and r such that $v \equiv w$ in Λ^*. Then $vt_i = wt_i$ and hence $\psi_v(R) = \psi_w(R)$. Thus, if x is any element of Λ^*, we can define a function $\psi_x : \mathcal{R} \to Z_2$, where $\psi_x R) = \psi_w(R)$ for all words w equivalent to x in Λ^*.

Now, if R is a reflection in G^* then $\varphi^{-1}(R)$ consists of certain reflections and certain glide transformations. The reflections are all those in geodesics that map to F_R. The glide transformations consist of compositions of some reflection in a geodesic in $\varphi^{-1}(R)$ and translation along the geodesic by some element of Γ. Let $\mathcal{R}_R$ be the set of reflections in $\varphi^{-1}(R)$. Let w is a word in p, q, and r, in Λ^* and $\overline{w}, \overline{p}, \overline{q}$, and $\overline{r}$ the corresponding elements in G^*. By lifting the path of triangles on Σ defined by $\overline{w}$ then the number of number of times $\overline{w}$ crosses F_R in Σ will be the number of times that w crosses any F_S in U, where $S \in \mathcal{R}_R$, i.e., $\sum_{S \in \mathcal{R}_R} \psi_w(S)$.

But Σ fails to split if and only if there is some word w equivalent to 1 in G^* which crosses F_R an odd number of times. A word is equivalent to 1 in G^* if and only if it is equivalent to an element $\Gamma \subseteq \Lambda^*$, so Σ fails to split if and only if there is some word w equivalent to an element in Γ which crosses mirrors corresponding to geodesics in $\mathcal{R}^R$ an odd number of times. We obtain.

Theorem 12 *The surface Σ splits at R if and only if $\sum\limits_{S \in \mathcal{R}_R} \psi_x(S) = 0$ for* *all* $x \in \Gamma$.

Constructing tilings that split is simply a matter of constructing normal subgroups of Λ^* which have this property. Since we are interested in tilings of surfaces of finite genus which split, we should also add the extra requirement that our normal subgroups have finite index in Λ^*. Unfortunately, Λ^* is a non-abelian infinite group with an infinite number of normal subgroups. Furthermore, checking every element of such a subgroup to see if it satisfies our condition is a difficult task, given that the number of such elements if infinite.

Since random generation of such normal subgroups seems impossible, we must find some other way of creating them. For instance, given a normal subgroup Γ which does not work, can we construct another normal subgroup which does? Suppose that Σ does not split along some mirror F_R. Then Γ must contain some element x such that $\sum_{S \in \mathcal{R}_R} \psi_x(S) \neq 0$. Our plan is simply to throw out these bad elements. We will do that in the next section, but before moving on we will prove a cocycle property of the parity functions.

Theorem 13 *Let notation be as above. Then*

$$\psi_{xy}(S) = \psi_x(S) + \psi_y(x^{-1}Sx) \tag{9}$$

for all $x, y \in \Lambda^*$ *and all* $S \in \mathcal{R}$.

Proof. We will proceed through a bizarre sort of induction on y. Our plan is to first show that $\psi_{xd}(S) = \psi_x(S) + \psi_d(x^{-1}Sx)$ whenever $x \in \Lambda^*$ and $d \in \{p, q, r\}$, and then to show that the formula works for (x, yd) whenever it works for (x, y).

Note first that $\psi_d(S) = \mathcal{X}_d(S) = \left\{ \begin{array}{ll} 1 & S = d \\ 0 & S \neq d \end{array} \right.$, since one path from the identify triangle to the d triangle is simply 1, d. Now, suppose that we are at the x triangle, and we cross the d-type edge. Then we have crossed over the xdx^{-1} mirror to the xd triangle, so $\psi_{xd} = \psi_x + \mathcal{X}_{xdx^{-1}}$. But $\mathcal{X}_{xdx^{-1}}(S) = \mathcal{X}_d(x^{-1}S_x) = \psi_d(x^{-1}Sx)$ for all $S \in \mathcal{R}$. Therefore, $\psi_{xd}(S) = \psi_x(S) + \psi_d(x^{-1}Sx)$ for all $S \in \mathcal{R}$.

Now suppose that $\psi_{xy}(S) = \psi_x(S) + \psi_y(x^{-1}Sx)$ for some $x, y \in \Lambda^*$and all $S \in \mathcal{R}$. Then

$$\begin{aligned}
\psi_{x(yd)}(S) &= \psi_{(xy)d}(S) \\
&= \psi_{xy}(s) + \psi_d(y^{-1}x^{-1}Sxy) \\
&= \psi_x(S) + \psi_y(x^{-1}Sx) + \psi_d(y^{-1}x^{-1}Sxy) \\
&= \psi_x(S) + \psi_{yd}(x^{-1}Sx).
\end{aligned}$$

Therefore, by induction $\psi_{xy}(S) = \psi_x(S) + \psi_y(x^{-1}Sx)$ for all $x, y \in \Lambda^*$ and $S \in \mathcal{R}$. ■

There is another way of looking at cocycle property given in the previous theorem. Consider the function $\vartheta : \Lambda^* \to S_{\mathcal{R}}$ (permutation group of $\mathcal{R}$) given by $\vartheta(x)(S) = x^{-1}Sx$. Then ϑ is a homomorphism, so we can define a wreath product $\Lambda^* \wr Z_2$ whose multiplication is given by $(g_1, \psi_1)(g_2, \psi_2) = (g_1, g_2, \psi_3)$, where $\psi_3(S) = \psi_1(S) + \psi_2(\vartheta(g_1)(S)) = \psi_1(S) + \psi_2(g_1^{-1}Sg_1)$. If $\psi_1 = \psi_{g_1}$ and $\psi_2 = \psi_{g_2}$, however, we get that $\psi_3 = \psi_{g_1 g_2}$. Therefore, the set of all (g, ψ_g) where $g \in \Lambda^*$, forms a subgroup of $\Lambda^* \wr Z_2$, isomorphic to Λ^*.

Notation By the way, for notational clarity, we will from now on write $g\psi_h(S)$ instead of $\psi_h(\vartheta(g)(S))$.

Remark 14 *Note that $\psi_{id} = 0$, so that $0 = \psi_{id} = \psi_{gg^{-1}} = \psi_g + g\psi_{g^{-1}}$. It follows that*

$$\psi_{g^{-1}} = g^{-1}\psi_g. \tag{10}$$

Using this formulae we further obtain:

$$\begin{aligned} \psi_{gtg^{-1}} &= \psi_g + g\psi_{tg^{-1}} \\ &= \psi_g + g(\psi_t + t\psi_{g^{-1}}) \\ &= \psi_g + g\psi_t + gtg^{-1}\psi_g. \end{aligned} \tag{11}$$

5 The Construction of Γ_R and $\widetilde{\Gamma}$.

Definition 15 *Let Γ_R be those elements x of Γ such that $\sum_{S \in \mathcal{R}_R} \psi_x(S) = 0$.*

Intuitively, Γ_R should be a subgroup of Γ, and it should be of index 2 (we will eventually prove these facts.) Unfortunately, it is not clear whether or not Γ_R is normal. We need to determine whether $\psi_{txt^{-1}}(S) = 0$ (for $t \in \Lambda^*$) when $\psi_x(S) = 0$. To do this, we need some way of calculating $\psi_{txt^{-1}}$ in terms of ψ_x. (Such a tool would also help us prove that $\Gamma_R \leq \Gamma$ and that $|\Gamma : \Gamma_R| = 2$.) The tool we need is given in equation 9 of Theorem 13.

Proposition 16 *Let notation be as above. Then Γ_R is a subgroup of Λ^*.*

Proof. Let $g, h \in \Gamma_R$. Then $\psi_{gh} = \psi_g + g\psi_h$. But $x \in \mathcal{R}_R \subseteq \varphi^{-1}(R)$ implies that $g^{-1}xg \in \varphi^{-1}(\varphi(g^{-1}xg)) = \varphi^{-1}\left(\varphi(g)^{-1}\varphi(x)\varphi(g)\right) = \varphi^{-1}(\varphi(x)) = \varphi^{-1}(R)$, since $\varphi(g) = 1$, by definition. Furthermore, $g^{-1}xg$ must be a reflection since x is, so $g^{-1}xg \in \mathcal{R}_R$. Therefore,$\vartheta(g)$ permutes the elements of $\mathcal{R}_R$.

So $\sum\limits_{x\in\mathcal{R}_R}(g\psi_h)(x)$ is even since $\sum\limits_{x\in\mathcal{R}_R}\psi_h(x)$ is. Thus, $\sum\limits_{x\in\mathcal{R}_R}(\psi_g+g\psi_h)(x) = \sum\limits_{x\in\mathcal{R}_R}\psi_g(x)+\sum\limits_{x\in\mathcal{R}_R}(g\psi_h)(x)$ is also even, so $gh\in\Gamma_R$ and Γ_R is a subgroup. ■

Proposition 17 *The index of* Γ_R *is given by* $[\Gamma:\Gamma_R]=1$ *if* Σ *splits at* F_R*, and* $[\Gamma:\Gamma_R]=2$ *otherwise.*

Proof: If Σ splits at F_R, then every closed path of triangles in Θ crosses F_R an even number of times. Thus, any path of triangles in Θ' from the identity to some element of Γ crosses geodesics in $\mathcal{R}_R$ an even number of times. Thus $\Gamma_R=\Gamma$, so $[\Gamma:\Gamma_R]=1$.

If Σ does not split at F_R, then there is some word w whose corresponding path of triangles in Θ is closed and crosses F_R an odd number of times. Let g be the element of Λ^* equivalent to w. Then $g\notin\Gamma_R$. Suppose that $h\in\Gamma$ but $h\notin\Gamma_R$. Then $\psi_{gh}=\psi_g+g\psi_h$. But $\vartheta(g)$ simply permutes the elements of $\mathcal{R}_R$, so $\sum\limits_{S\in\mathcal{R}_R}g\psi_h(S)=\sum\limits_{S\in\mathcal{R}_R}\psi_h(S)=1$, and so

$$\sum_{S\in\mathcal{R}_R}\psi_{gh}(S)=\sum_{S\in\mathcal{R}_R}\psi_g(S)+\sum_{S\in\mathcal{R}_R}g\psi_h(S)=1+1=0.$$

Therefore, $gh\in\Gamma_R$ whenever $h\in\Gamma$ and $h\notin\Gamma_R$, so $[\Gamma:\Gamma_R]\leq 2$. But $g\in\Gamma\backslash\Gamma_R$, so $[\Gamma:\Gamma_R]\neq 1$, and so $[\Gamma:\Gamma_R]=2$. ■

Unfortunately, Γ_R need not be normal as we see in the following example.

Example 18 *Let* $(l,m,n)=(5,5,5)$ *so that* $\Lambda^*=\langle p,q,r\mid p^2,q^2,r^2,(pq)^5,(qr)^5,(rp)^5\rangle$ *and let* Γ *be the smallest normal subgroup containing* $rpqp$*. Then* $G^*=\Lambda^*/\Gamma\cong D_5$*, and* G^* *is the tiling group of a* $(5,5,5)$ *tiling on a surface of genus* 2*. Let* $R=q$*. Then* $rpqp\in\Gamma_q$*, but* $(pq)(rpqp)(pq)\notin\Gamma_q$*, so* Γ_q *is not normal.*

Since Γ_R is not necessarily normal, we cannot take the quotient of Λ^* by Γ_R to find a tiling which splits. (Note that a covering of tiling $\Sigma_R = U/\Gamma_R\longrightarrow U/\Gamma\longrightarrow\Sigma$ exists but not every edge generates a reflection on the covering surface, i.e., it is not kaleidoscopic.) All hope is not lost though: we can take the quotient of Λ^* by the *core* of Γ_R. Hopefully, this will be a tiling of a surface of finite genus which splits along R.

Let $\widetilde{\Gamma}$ be the core of Γ_R. Then $\widetilde{\Gamma}$ is the intersection of all $t\Gamma_R t^{-1}$, where $t\in\Lambda^*$. To find a better description of $\widetilde{\Gamma}$, we again turn to our parity functions.

Proposition 19 *Let $g \in \Lambda^*$, and let $S = \varphi(g)R\varphi(g)^{-1}$. Then $g\Gamma_R g^{-1} = \Gamma_S$.*

Proof: If $x \in \mathcal{R}_S$,then $g^{-1}xg \in \varphi^{-1}(\varphi(g^{-1}xg)) = \varphi^{-1}\left(\varphi(g)^{-1}S\varphi(g)\right) = \varphi^{-1}(R)$. But $g^{-1}xg$ is a reflection since x is, so $g^{-1}xg \in \mathcal{R}_R$. Therefore, $\vartheta(g)(\mathcal{R}_S) \subseteq \mathcal{R}_R$. But $\vartheta(g)\vartheta(g^{-1})$ is the identity map and $\vartheta(g^{-1})(\mathcal{R}_R) \subset \mathcal{R}_S$, so $\vartheta(g)$ maps $\mathcal{R}_S$ onto $\mathcal{R}_R$.

If $t \in \Gamma_R$, then $\psi_{gtg^{-1}} = \psi_g + g\psi_t + gtg^{-1}\psi_g$ by formula 11. But

$$\sum_{T\in\mathcal{R}_S} g\psi_t(T) = \sum_{T\in\mathcal{R}_S} \psi_t(g^{-1}Tg) = \sum_{T\in\mathcal{R}_R} \psi_t(T),$$

which is zero since $t \in \Gamma_R$. Also, $\varphi(gtg^{-1}) = \varphi(g)\varphi(g)^{-1} = 1$, so conjugation by gtg^{-1} simply permutes the elements of $\mathcal{R}_S$. Therefore,

$$\sum_{T\in\mathcal{R}_S} gtg^{-1}\psi_g(T) = \sum_{T\in\mathcal{R}_S} \psi_g(T),$$

so

$$\sum_{T\in\mathcal{R}_S} (\psi_g + gtg^{-1}\psi_g)(T) = 0$$

Therefore,

$$\sum_{T\in\mathcal{R}_S} \psi_{gtg^{-1}}(T) = \sum_{T\in\mathcal{R}_S} \left(\psi_g + g\psi_t + gtg^{-1}\psi_g\right)(T) = 0,$$

so $gtg^{-1} \in \Gamma_S$. ■

Corollary 20 *The subgroup $\widetilde{\Gamma}$ is the set of all $g \in \Lambda^*$ so that $\sum_{T\in\mathcal{R}_S} \psi_g(T) = 0$ for all S conjugate to R in G^*.*

6 Properties of $\widetilde{G}^*$

Let $\widetilde{G}^* = \Lambda^*/\widetilde{\Gamma}$, and let $I = \Gamma/\widetilde{\Gamma}$. Then $G^* = \widetilde{G}^*/I$. Now let $\mu : \widetilde{G}^* \to G^*$ and $\widetilde{\varphi} : \Lambda^* \to \widetilde{G}^*$ be the proper homomorphisms. Let $\widetilde{p}, \widetilde{q}$ and $\widetilde{r}$ be the three generating reflections of $\widetilde{G}$. We claim that $\widetilde{G}^*$ tiles a surface of finite genus.

Consider the covering map of U onto Σ. What is the preimage of a point in Σ? Triangles $t_t, t_2 \in \ominus'$ map to the same triangle in Σ if and only if $t_1 \in \Gamma t_2$. Therefore, we would expect that s_1 and s_2 map to the same place in Σ if and only if $s_1 \in \Gamma s_2$. Assuming this is true, we could have constructed Σ as the set of all Γ-orbits of U under the quotient topology.

Let $\widetilde{\Sigma}$ be the set of all $\widetilde{\Gamma}$-orbits of U, where the topology on $\widetilde{\Sigma}$ is the quotient topology derived from U. Then the map $\pi : \widetilde{\Sigma} \to \Sigma$ given by $\pi\left(\widetilde{\Gamma} s\right) = \Gamma s$ is a covering map, so $\widetilde{\Sigma}$ is a surface covering Σ. Furthermore, the geodesic, kaleidoscopic tiling $\ominus$ on Σ lifts to a geodesic, kaleidoscopic tiling $\widetilde{\ominus}$ on Σ. Finally, if $g \in \widetilde{G}^*$, then we can associate with g the transformation on $\widetilde{\Sigma}$ given by $g\left(\widetilde{\Gamma} s\right) = \left(\widetilde{\varphi}^{-1}(g)\right) s$. The set of transformations associated with elements of $\widetilde{G}^*$ forms a group, which is in fact the tiling group of $\widetilde{\ominus}$.

Now that we have constructed our tiling of our new surface, we would like to check if it splits. Unfortunately, we constructed $\widetilde{G}^*$ as a quotient of two infinite groups, and find ourselves unable to do any computations. We need a better representation of $\widetilde{G}^*$.

Suppose that g_1 and g_2 are two distinct elements of $\widetilde{G}^*$, but that $\mu(g_1) = \mu(g_2)$. How can we distinguish between g_1 and g_2? By Corollary 20, g_1 and g_2 differ in the following ways: if we take $h_1 \in \widetilde{\varphi}^{-1}(g_1)$ and $h_2 \in \widetilde{\varphi}^{-1}(g_2)$, then $\sum_{T \in \mathcal{R}_S} \psi_{h_1}(T) \neq \sum_{T \in \mathcal{R}_S} \psi_{h_2}(T)$ for some reflection S conjugate to R in G^*.

Therefore, we can label any element g of $\widetilde{G}^*$ using $\mu(g)$ and the sequence of $\sum_{T \in \mathcal{R}_S} \psi_h(T)$, where h is any element of $\widetilde{\varphi}^{-1}(g)$ and S ranges over everything in the conjugacy class of R. More formally, let Π be the conjugacy class of R in G^*, and define $\tau : \widetilde{G}^* \to Z_2^{\Pi}$ by $\tau_g(S) = \sum_{T \in \mathcal{R}_S} \psi_h(T)$, where $h \in \widetilde{\varphi}^{-1}(g)$. Then we can label any element g of $\widetilde{G}^*$ by the ordered pair $(\mu(g), \tau_g)$.

There is another way of looking at the parity functions τ. If w is a word in p, q, and r, define $\tau_w : \Pi \to Z_2$ to be the mirror-crossing parity function for mirrors in Π. (We are doing the same thing we did to define ψ, but we are doing it in Σ instead of U.) Suppose that $g \equiv w$ in $\widetilde{G}^*$, so that $w \equiv h$ in Λ^* for some $h \in \widetilde{\varphi}^{-1}(g)$. Then $\tau_g(S) = \sum_{T \in \mathcal{R}_S} \psi_h(T) = \sum_{T \in \mathcal{R}_S} \psi_w(T) = \tau_w(S)$. Therefore, τ_g is the parity function of the path in G^*corresponding to some g-equivalent word in $\widetilde{G}^*$.

Since we defined the τ parity functions and the ψ parity functions in the same way, many of the results for ψ parity functions apply to the τ parity functions. Specifically, if v and w are words, then $\tau_{vw} = \tau_v + v\tau_w$. Therefore, if $x, y \in \widetilde{G}^*$, then $\tau_{xy} = \tau_x + \mu(x)\tau_y$ (note that $\mu(x)$ and x are represented by the same word. We write $\mu(x)\tau_y$ instead of $x\tau_y$ since the domain of τ_y lies in G^* and not $\widetilde{G}^*$.) As before, we can then describe $\widetilde{G}^*$ as a subgroup of $G^* \wr Z_2$. Specifically, $\widetilde{G}^* = \langle \widetilde{p}, \widetilde{q}, \widetilde{r} \rangle$, where$\widetilde{p} = (p, \mathcal{X}_p), \widetilde{q} = (q, \mathcal{X}_q)$, and $\widetilde{r} = (r, \mathcal{X}_r)$. (Note that p, for instance, is not guaranteed to be in II, so $\mathcal{X}_p$ may be the constant zero function.)

Based on what we have done, one might think that $\widetilde{\Sigma}$ must split at $F_{\widetilde{R}}$ for any reflection $\widetilde{R}$ in $\mu^{-1}(R)$. Unfortunately this is not the case. Let us apply the criterion in Theorem 12: $\widetilde{\Sigma}$ splits at a mirror S if and only if $\sum\limits_{T \in \mathcal{R}_S} \psi_g(T) = 0$ for all $g \in \widetilde{\Gamma}$. What guarantee do we have that $\sum\limits_{T \in \mathcal{R}_S} \psi_g(T) = 0$ for all $g \in \widetilde{\Gamma}$? We know that $\sum\limits_{T \in \mathcal{R}_R} \psi_g(T) = 0$ for all $g \in \widetilde{\Gamma}$, but $\mu^{-1}(R)$ may contain several reflections, so $\mathcal{R}_{\widetilde{R}}$ is not necessarily equal to $\mathcal{R}_R$.

Fortunately, it is often the case that $\widetilde{\Sigma}$ does split. Through experimentation, it appears that surfaces constructed in this manner actually split much more often than random surfaces and, as we shall see in the next section, we already know certain properties of this splitting.

7 Splitting Properties of $\widetilde{\Sigma}$

Our first result is a fundamental theorem which will be the basis for all other theorems in this section. We constructed $\widetilde{\Sigma}$ specifically so that this theorem would hold, and it explains the purpose of all the work we have done thus far:

Theorem 21 *Let* $F_{\mu^{-1}(R)} = \bigcup\limits_{S \in \mu^{-1}(R)} F_S$. *Then* $\widetilde{\Sigma}$ *splits at* $F_{\mu^{-1}(R)}$. *Furthermore, g and h correspond to triangles in the same component of* $\widetilde{\Sigma} \backslash F_{\mu^{-1}(R)}$ *only if* $\tau_g(R) = \tau_h(R)$.

Proof. First note that $\tau_i(R) = 0$ and that $\tau_x(R) = 1$ for at least one $x \in \widetilde{G}^*$. We shall show that the only way to change $\tau(R)$ is to cross a mirror corresponding to a reflection in $\mu^{-1}(R)$.

Let $g \in \widetilde{G}^*$, and let $h = g\widetilde{d} = Sg$, where $\widetilde{d} \in \{\widetilde{p}, \widetilde{q}, \widetilde{r}\}$ and $S = g\widetilde{d}g^{-1}$ is the proper reflection in $\widetilde{G}^*$. Assume that $\tau_g(R) \neq \tau_h(R)$. We want to show that $\mu(S) = R$. Since $\widetilde{d} = (d, \mathcal{X}_d)$, $\tau_h = \tau_g + \mu(g)\mathcal{X}_d$. But $\tau_h(R) \neq \tau_g(R)$, so $\mathcal{X}_{\mu(g)d\mu(g)^{-1}}(R) = (\mu(g)\mathcal{X}_d)(R) = \psi_g(d) - \psi_h(d) = 1$. Therefore, $R{=}\mu(g)d\mu(g)^{-1} = \mu\left(g\widetilde{d}g^{-1}\right) = \mu(S)$. ■

Conjecture 22 *The set difference* $\widetilde{\Sigma} \backslash F_{\mu^{-1}(R)}$ *has exactly two components.*

Not only does $\widetilde{\Sigma}$ split at $F_{\mu^{-1}(R)}$,but we know exactly what the components of $\widetilde{\Sigma}\backslash F_{\mu^{-1}(R)}$ are! Conjecture 22 makes the relationship between $\tau(R)$ and components an if and only if relationship, since it leaves no room for more than one $\tau(R) = 0$ or $\tau(R) = 1$ component.

Of course, $\mu^{-1}(R)$ may consist of several reflections, so this theorem does not force $\widetilde{\Sigma}$ to split along any one mirror $\widetilde{R} \in \mu^{-1}(R)$. However, there is one case where $\widetilde{\Sigma}$ must split along $F_{\widetilde{R}}$.

Corollary 23 *Suppose that $\widetilde{R}$ is the only reflection in $\mu^{-1}(R)$. Then $\widetilde{\Sigma}$ splits at $F_{\widetilde{R}}$.*

If Conjecture 22 holds, then the converse is also true. Even if Conjecture 22 is false, examples where $\widetilde{\Sigma}$ splits at $F_{\widetilde{R}}$and $\widetilde{R}$ is not the only reflection in $\mu^{-1}(R)$ must be extremely rare, so Corollary 23 still provides a good test for splitting when trying to generate tilings that split.

The following theorem provides a very simple way of showing that $\widetilde{\Sigma}$ splits:

Theorem 24 *If $|\ker \mu| = 2$, then $\widetilde{\Sigma}$ splits at $F_{\widetilde{R}}$.*

Proof. Assume that $\widetilde{\Sigma}$ does not split at $F_{\widetilde{R}}$. Since $|\ker \mu| = 2$, $\left|\widetilde{G}^*\right| = 2\,|G^*|$, so $\widetilde{\Sigma}$ splits into exactly two components at $F_{\mu^{-1}(R)}$. Moveover, each component is of size $|G^*|$,so there is exactly one preimage of every triangle of G^* in each component. Let S be the element of $\mu^{-1}(R)$ that is not $\widetilde{R}$. Since $\widetilde{\Sigma}$ does not split at $F_{\widetilde{R}}$, we must be able to move between the two components of $\widetilde{\Sigma}\backslash F_{\mu^{-1}(R)}$ without crossing $F_{\widetilde{R}.}$ Therefore, there is some $a \in \widetilde{G}^*$ and $\widetilde{d} \in \left\{\widetilde{p}, \widetilde{q}, \widetilde{r}\right\}$ such that $\tau_a(R) = 0, \tau_{a\widetilde{d}}(\mathcal{R})$ =1, and $a\widetilde{d}a^{-1} \neq \widetilde{R}$. However, $\mu\left(a\widetilde{d}a^{-1}\right) = R$, so $a\widetilde{d}a^{-1}$ must be S. Therefore, $a\widetilde{d} = Sa$, so $\tau_{Sa}(\mathcal{R}) = 1$. We claim that $\tau_S(\mathcal{R}) = 1$. Since $\tau_{\widetilde{R}}(R) = 1$, we will get that S and $\widetilde{R}$ are in the same component of $\widetilde{\Sigma}\backslash F_{\mu^{-1}(R)}$, a contradiction, since each component of $\widetilde{G}^*$ contains exactly one copy of everything in G^*.

We know that $\tau_{Sa}(R) = \tau_S(R) + \mu(S)\tau_a(R)$. But $\mu(S) = R$, so $\mu(S)\tau_a(R) = R\tau_a(R) = \tau_a(R^{-1}RR) = \tau_a(R)$. Thus, $\tau_S(R) = \tau_{Sa}(R) - \mu(S)\tau_a(R) = \tau_{Sa}(R) - \tau_a(R) = 1 - 0 = 1$. ■

By the way, the converse of Theorem 24 is false, as the following example shows.

Example 25 *Let* $G^* = (Z_4 \times Z_4) \times Z_2$*. Then* G^* *is the tiling group of a tiling of a genus 3 surface* Σ *(where* $|pq| = |qr| = |rp| = 4$*) which does not split at* F_p*. The resulting* $\widetilde{\Sigma}$ *is a genus 9 which splits at* $F_{\widetilde{p}}$*, but* $|\ker \mu| = 4$.

The following theorem give us another group theoretic characterization of splitting.

Theorem 26 *The surface* Σ *splits at* F_R *if and only if* $\left|\widetilde{G^*}\right| = G^*$.

Proof. 'Suppose that $\widetilde{\Sigma}$ splits at F_R. If $t \in \Gamma$, then $\sum_{S \in \mathcal{R}_R} \psi_t(S)$ must therefore be even, so that $\Gamma_{\mathcal{R}} = \Gamma$. But Γ is normal in Λ^*, so $\widetilde{\Gamma} = \text{core}(\Gamma_R)$ $= \text{core}(\Gamma) = \Gamma$. Therefore, $\widetilde{G^*} = \Lambda^*/\widetilde{\Gamma} = \Lambda^*/\Gamma = G^*$. Now suppose that $\widetilde{G^*} = G^*$. Then $\Lambda^*/\Gamma = \Lambda^*/\widetilde{\Gamma}$, so $\Gamma = \widetilde{\Gamma}$. But $\widetilde{\Gamma} \leq \Gamma_R \leq \Gamma$, so $\Gamma = \Gamma_R$. Therefore, $\sum_{S \in \mathcal{R}_R} \psi_t(S)$ is even for all $t \in \Gamma$, so G^* splits at F_R. ∎

Corollary 27 *The surface* Σ *splits at* F_R *if and only if* $\widetilde{G^*} = |G^*|$.

Even though Corollary 27 seems somewhat pointless and trivial, it actually accomplishes one of our stated goals. Recall that is no known fully group-theoretic criterion to determine whether a tiling splits. However, Theorem 13 is just that: it decides whether a tiling splits based on the order of a certain subgroup of $G^* \wr Z_2$.

8 Further Questions

All attempt at analyzing separability more closely has resulted in more questions than answers. Here is a list of questions that are promising avenues of attack on the separability question.

1. Find a fast algorithm that based on the ideas of Theorem 6 for deciding separability. For instance if one can build a path from t_i to pt_i that does not pass through F_p then p is not separating.

2. See if there are any efficient graph-theoretic algorithms that determine separability using the dual graph. The algorithm in Theorem 6 is analogous to a breadth first search.

3. Is there a bound on the size of $[\widetilde{G^*} : G^*]$.

4. What is the structure of $\ker \mu$?
5. Can $\ker \mu \longrightarrow \widetilde{G}^* \longrightarrow G^*$ be identified with any of the homology covers described in [1]?
6. Is there a quick way to compute $\widetilde{G}^*$.
7. Prove or provide a counterexample to Conjecture 22

References

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