

# THE GALOIS CORRESPONDENCE FOR BRANCHED COVERING SPACES AND ITS RELATIONSHIP TO HECKE ALGEBRAS

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ABSTRACT. There is a very beautiful correspondence between branched covers of the Riemann sphere  $\mathbb{P}^1$  and subgroups of the fundamental group  $\pi_1(\mathbb{P}^1 - \{\text{branch points}\})$ , exactly analogous to the correspondence between subfields of an algebraic extension  $E/F$  and subgroups of the Galois group  $Gal(E/F)$ . This paper explores the concept of a Hecke algebra, which in this context is a generalization of the Galois group to the case of non-Galois covers  $S/\mathbb{P}^1$ . Specifically, we show that the isomorphism type of a Hecke algebra  $\mathbb{C}[H \backslash G / H]$  is completely determined by the decomposition of the induced character  $1_H^G$ , and that the character of the homology representation of a Galois group generalizes to one for Hecke algebras, the decomposition of which depends on certain double cosets in the group corresponding to the Galois closure of the cover  $S/\mathbb{P}^1$ .

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## 1. THE GALOIS CORRESPONDENCE FOR COVERING SURFACES

The basic reference for the material in this section is [9]. See also [3]. Throughout, we let  $\mathbb{P}^1$  denote the Riemann sphere.

We first recall the notion of a covering surface. Given two topological surfaces  $X$  and  $Y$ , we say that  $Y$  is a covering surface of  $X$  if there exists a continuous surjective map  $p$  from  $Y$  to  $X$ , and each point in  $X$  has a neighborhood  $V$  such that  $p^{-1}(V)$  breaks up into a disjoint union of open sets, each of which is homeomorphic to  $V$  under  $p$ . Intuitively,  $p$  “wraps”  $Y$  onto  $X$ , as one can see in the case where  $\mathbb{R}^2$  covers the torus upon modding out by  $\mathbb{Z} \times \mathbb{Z}$ .

We shall consider covering surfaces which have the additional property of being *analytic manifolds*, or *Riemann surfaces*. This means that each point on the surface has a neighborhood conformally equivalent to the complex disk (see [9], Ch. 19 or [8], Ch. 1).

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We also insist that our covers be compact and connected (and thus finite-sheeted), so that we have only a finite number of *ramification points*, or points where the local coordinate map is given by  $f = z^e g(z)$ , where  $g(z)$  is analytic and non-vanishing at the origin, and  $e > 1$ . When such a Riemann surface  $Y$  covers another Riemann surface  $X$  via  $p$ , the images of the ramification points under  $p$  are called the *branch points* of  $X$ .

For these so-called *branched covers*  $Y \xrightarrow{p} X$ , let  $B$  be the set of branch points and let  $X^\circ = X_B^\circ = X - B$  denote the punctured surface obtained by removing the branch points from  $X$  and let  $Y^\circ = p^{-1}(X^\circ)$ . One can fix base points  $x_0 \in X^\circ$  and  $y_0 \in Y^\circ$ , and define a group action of  $\pi_1(X^\circ, x_0)$  on the pre-image of  $x_0$  (called the *fiber* of  $x_0$ ), simply by lifting a loop  $\gamma$  in  $\pi_1(X^\circ, x_0)$  to the various paths which begin and end on the points of the fiber of  $x_0$  (see, e.g., [14]). In the case of *normal* or *Galois* covers, this action is transitive on  $p^{-1}(x_0)$ , and the action can be extended to a full covering transformation of  $Y/X$ . This group of covering transformations is called the *Galois group of  $Y/X$* , and is denoted  $Gal(Y/X)$ .

Now the motivation for dressing these terms in the language of Galois theory is the following:

**Proposition 1** (Galois Correspondence). Let  $X$  be a compact connected Riemann surface,  $B \subset X$  a finite set,  $X_B^\circ = X - B$ ,  $\widetilde{X}_B^\circ$  the universal cover of  $X_B^\circ$ , and fix base points  $x_0 \in X_B^\circ$  and  $\widetilde{x}_0 \in \widetilde{X}_B^\circ$ . Then there is an inclusion-reversing bijection between the subgroups of finite index in  $\pi_1(X_B^\circ, x_0)$  and (topological equivalence classes of) the compact, branched covers  $P : Y \rightarrow X$  with branch set in  $B$  such that:

- For each subgroup of finite index  $H \subset \pi_1(X_B^\circ, x_0)$  there is a unique (up to topological equivalence) covering space  $p_H : Y_H^\circ \rightarrow X_B^\circ$  lying between  $X_B^\circ$  and  $\widetilde{X}_B^\circ$ , such that for any  $y_H \in p_H^{-1}(x_0)$ ,  $\pi_1(Y_H^\circ, y_H) \cong H$ . Moreover,  $p_H : Y_H^\circ \rightarrow X_B^\circ$  may be completed to a compact covering  $p_H : Y_H \rightarrow X$ , branched over  $B$ .
- Conversely, for each compact, connected, branched covering  $p : Y \rightarrow X$  whose branch points lie in  $B$  and  $y \in p^{-1}(x_0)$ ,  $\pi_1(Y^\circ, y)$  is isomorphic to a subgroup  $H$  of  $\pi_1(X_B^\circ, x_0)$ , and  $Y \cong Y_H$ .
- Furthermore, Galois covers  $Y/X$  correspond exactly to normal subgroups  $N$  of  $\pi_1(X_B^\circ, x_0)$ , for some set  $B$ , in which case  $Gal(Y/X) \simeq \pi_1(X_B^\circ, x_0)/N$  and  $X \cong Y/N$ .

This inclusion-reversing bijection is similar to that seen in classical Galois theory, where one has a (finite) extension of fields  $E/F$ , and one speaks of a correspondence between the subgroups of  $E$ -automorphisms which fix  $F$  and the various intermediate fields between  $E$  and  $F$ . In fact one can interpret the above proposition in terms of field theory by viewing Riemann surfaces as the zero-set of irreducible bi-variate polynomials  $f(Z, W) \in \mathbb{C}[Z, W]$ , in which case the intermediate covers become finite extensions of the function field  $\mathbb{C}(z)$ , and the covering transformations become field automorphisms which permute the roots of  $f(z, W)$ , for any fixed  $z \in \mathbb{C}$ . See [12] or [8] for more details.

Just as in classical Galois theory, this topological Galois correspondence proves to be a valuable tool for understanding a lattice of covering spaces because one is working with concrete groups. Though for general  $Y/X$ ,  $\pi_1(Y^\circ, y)$  is infinite, one need only look at its finite homomorphic images to fully understand the groups of covering transformations associated with its finite-sheeted covering spaces. Indeed for these spaces, at all but finitely many points  $x_0$  of  $X$ ,  $G = Gal(Y/X)$  permutes simply transitively the fiber of  $x_0$ . On these fibers the left action of  $G$  is the same as the left regular representation of  $G$ . This is a consequence of the fact that all but finitely many points in  $X$  are unramified (recall that

we assume  $Y$  is compact, so this is forced). At the branch points  $Q$  there are non-trivial cyclic stabilizers for all the ramified points  $P_i$  above  $Q$  (see [1]). The number and order of these stabilizers play a significant role in determining the structure of  $Y$  as well as the homology representation of  $G$  (see Section 5).

## 2. HECKE ALGEBRAS AND SOME OF THEIR BASIC PROPERTIES

The above discussion should indicate that the optimal covering space  $Y/X$  one could hope for is a Galois one. In this case one has complete Galois correspondence between the subgroups of  $\text{Gal}(Y/X)$  and the intermediate covers between  $Y$  and  $X$ .

Unfortunately, not all covers are Galois. The goal of this section is to describe a generalization of the Galois group to the case of arbitrary covers, not necessarily Galois. This generalization, known as a Hecke algebra, allows one to recover much of the geometric information about  $Y$ . It also possesses a representation theory which generalizes that for Galois groups.

For the basic definitions and structure theorems concerning algebras, see [5] or [6].

For a finite group  $G$ , and  $H$  a subgroup of  $G$ , one defines the *Hecke algebra of  $G$  with respect to  $H$*  to be the subalgebra  $\mathbb{C}[H \backslash G / H]$  of  $\mathbb{C}[G]$  with basis elements the double coset averages

$$(1) \quad \epsilon_H = \frac{1}{|H|} \sum_{h \in H} h,$$

where the  $g \in D$  are selected to give the partition of  $G$  in to double cosets:

$$G = \bigcup_{g \in D} HgH.$$

For the proof that  $\mathbb{C}[H \backslash G / H]$  forms a subalgebra of  $\mathbb{C}[G]$ , see [3].

One should begin by observing that the Hecke basis elements  $\epsilon_H * g * \epsilon_H$  correspond naturally to the double cosets  $HgH \subset G$ . Thus the dimension of  $\mathbb{C}[H \backslash G / H]$  is exactly the number of  $H$ -double cosets in  $G$ . Further, we can view  $HgH$  (as well as  $\epsilon_H g \epsilon_H$ ) as the  $H$ -orbit of the coset  $gH$ , where  $H$  acts on  $G/H$  by left multiplication. That is,

$$(2) \quad HgH = \{hgH | h \in H\} = \bigcup_{i=1}^s g_i H,$$

for some  $s$  elements  $g_i$  forming a left transversal of  $H$  in  $HgH$ . The analogous equation in the group algebra is the decomposition of a Hecke double coset average into a linear combination of coset averages:

$$(3) \quad \epsilon_H g \epsilon_H = \sum_{i=1}^s \frac{1}{s} g_i \epsilon_H,$$

where the normalization factor  $1/s$  appears because under the trivial representation both sides are 1.

Under this decomposition, the multiplication of two Hecke basis elements can be computed as follows. Since

$$\epsilon_H g \epsilon_H = \sum_{i=1}^s \frac{1}{s} g_i \epsilon_H$$

then for any other Hecke basis element  $\epsilon_H \tilde{g} \epsilon_H$ ,

$$\begin{aligned}
(4) \quad (\epsilon_H \tilde{g} \epsilon_H)(\epsilon_H g \epsilon_H) &= (\epsilon_H \tilde{g})(\epsilon_H g \epsilon_H) \\
&= \sum_{i=1}^s \frac{1}{s} \epsilon_H \tilde{g} g_i \epsilon_H \\
&= \sum_{g \in H/GH} \frac{s_g}{s} \epsilon_H g \epsilon_H
\end{aligned}$$

where  $s_g = |\{i : \tilde{g} g_i \in HgH\}|$ , and  $H = \bigcup_i g_i H$ .

With this description, we can give the following interpretation to the Hecke algebra. Suppose we have some representation of  $\mathbb{C}[G]$  on a finite dimensional vector space  $V$ . The subspace  $V^H$  of  $H$ -invariants is just  $\epsilon_H V$ . When  $H \triangleleft G$ , we get a representation of  $G/H$  on  $\epsilon_H V$  simply by left multiplication. When  $H$  is not normal in  $G$ , however, this action is not well-defined. But we can still enlarge the  $H$ -action on  $\epsilon_H V$  via  $\mathbb{C}[H \backslash G / H]$ . To see that the action of  $\mathbb{C}[H \backslash G / H]$  by left multiplication stabilizes  $\epsilon_H V$ , simply observe that for any Hecke basis element  $\epsilon_H g \epsilon_H$ ,

$$\begin{aligned}
(\epsilon_H g \epsilon_H)(\epsilon_H V) &= \\
(\epsilon_H g \epsilon_H^2) V &= \\
(\epsilon_H g \epsilon_H) V &= \\
(\epsilon_H g)(\epsilon_H V) &
\end{aligned}$$

which is invariant under  $H$  since  $h \epsilon_H g \epsilon_H V = \epsilon_H g \epsilon_H V, \forall h \in H$ . Hence  $\mathbb{C}[H \backslash G / H]$  stabilizes  $\epsilon_H V$ .

More specifically, for any  $\epsilon_H v \in \epsilon_H V$ ,  $\epsilon_H g \epsilon_H \epsilon_H v = \epsilon_H g \epsilon_H v$  represents the  $HgH$ -orbit of  $v$ . The multiplication of two Hecke basis elements  $\epsilon_H g \epsilon_H$  and  $\epsilon_H \tilde{g} \epsilon_H$  corresponds to the permutation action of  $\tilde{g}$  on the  $H$ -orbits partitioning  $HgHv$ .

### Example 1.

Here is an example of the above statements. Consider the 3-dimensional Hecke algebra determined by  $G = D_{2,8,3} = \langle x, y | x^2 = y^8 = 1, xyx^{-1} = y^3 \rangle$  and  $H \cong D_4$ . There are three double coset averages,  $\epsilon_H 1 \epsilon_H$ ,  $\epsilon_H g_1 \epsilon_H$ , and  $\epsilon_H g_2 \epsilon_H$ , for some  $g_1, g_2 \in G$ . One of the double cosets has size 8, yielding say,  $\epsilon_H g_1 \epsilon_H$ , while the other two have size  $|H| = 4$ . Below is the table of structure constants for the basis element  $\epsilon_H g_1 \epsilon_H$ :

	1	$g_1$	$g_2$
1	0	1	0
$g_1$	1/2	0	1/2
$g_2$	0	1	0

Each row of the table indicates where  $\epsilon_H g_1 \epsilon_H$  sends the  $H$ -cosets averages in the corresponding double coset average. For instance, the values in the middle row indicate that  $\epsilon_H g_1 \epsilon_H$  sends one  $H$ -coset average of  $\epsilon_H g_1 \epsilon_H$  into  $\epsilon_H$ , the other into  $\epsilon_H g_2 \epsilon_H$ . The top and bottom row indicate that  $\epsilon_H g_1 \epsilon_H$  sends the remaining  $H$ -coset averages  $\epsilon_H 1 \epsilon_H$  and  $\epsilon_H g_2 \epsilon_H$  back into  $\epsilon_H g_1 \epsilon_H$ .

Perhaps the most important property of a Hecke algebra  $\mathbb{C}[H \backslash G / H]$  is that it is *semi-simple*. That is, any representation  $\rho$  of  $\mathbb{C}[H \backslash G / H]$  can be decomposed into a direct sum of irreducible ones. This follows from the semi-simplicity of the group algebra (see [5], Ch. 5).

## 3. ELEMENTS OF REPRESENTATION THEORY

This section outlines two important concepts in representation theory, that of induced representations and Frobenius Reciprocity.

First recall that for any representation  $\rho$  of  $\mathbb{C}[G]$  on a vector space  $V$ , and any subgroup  $H$  in  $G$ , we get a representation of  $\mathbb{C}[H]$  on  $V$  by restriction. We usually denote this restricted representation by  $\rho_H$  when the ambient group  $G$  is understood from the context.

We would now like to go the other direction, that is, given a representation  $\psi$  of  $\mathbb{C}[H]$  on  $W$ , we would like to get a representation of  $\mathbb{C}[G]$  on some space  $V$  associated with  $W$ . To do so, take any set  $\{g_\sigma\}$  of forming a left transversal for  $H$  in  $G$ . Then form the vector space

$$V = \bigoplus_{\sigma \in G/H} W^{g_\sigma}$$

where  $W^{g_\sigma}$  is the vector space isomorphic to  $W$  consisting of elements of the form  $g_\sigma \cdot w, w \in W$ . One can think of  $W^{g_\sigma}$  as just another copy of  $W$  with elements “labelled” by  $g_\sigma$ .

We let  $\mathbb{C}[G]$  act on  $V$  as follows: for  $g \in G$ , and  $g_\sigma \in T$ , write  $g \cdot g_\sigma = g_\tau h_{g,\tau}$ , for some unique  $h_{g,\tau} \in H$ . Then let  $g$  act on  $V$  by

$$g \cdot V = \bigoplus_{\tau \in G/H} (h_{g,\tau} \cdot W)^{g_\tau}$$

which maps  $V$  to  $V$ .

It is easy to check that this is a well-defined action independent of the choice of coset representatives (see [10]).

Another, more sophisticated way of viewing induced representations (which will be adopted later in this report), is in terms of tensor products. For the definition and properties of tensor products, see [6] or [5]. The tensor product allows us to view the induced representation of  $\mathbb{C}[H]$  on  $W$  just as a change of rings, or an “extension of scalars.” According to the theory of tensor products, given a subring  $R$  of  $S$ , and a left  $R$ -module  $M$ , we can construct a left  $S$ -module extending  $M$  via  $S \otimes_R M$ . In our case,  $R = \mathbb{C}[H]$ ,  $S = \mathbb{C}[G]$ , and  $M = W$ . Hence  $\psi^G = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$ .

Perhaps the simplest example of induced representations is  $1_H^G$ , the induced representation of the trivial representation on  $\mathbb{C}[H]$ . Here we simply get the permutation action of  $G$  on the  $H$ -coset space.

Induced representations are useful ways of constructing representations on  $\mathbb{C}[G]$  given ones on  $\mathbb{C}[H]$ . Usually we would like to know how  $\psi^G$  decomposes into  $G$ -irreducibles. Such knowledge can be obtained by the following (see [6]):

**Proposition 2** (Change of Rings Formula). Let  $R \subseteq S$  be rings,  $A$  an  $(S, R)$  bi-module,  $B$  a left  $S$  module, and  $C$  a left  $R$ -module. Then

$$(5) \quad \text{Hom}_S(A \otimes_R C, B) \cong \text{Hom}_R(C, B|_R).$$

With  $H$  a subgroup of  $G$ , letting  $R = \mathbb{C}[H]$ ,  $S = A = \mathbb{C}[G]$ ,  $C$  a  $\mathbb{C}[H]$ -representation, and  $B$  a  $\mathbb{C}[G]$ -representation, this result reduces to the following:

**Corollary 1** (Frobenius Reciprocity).

$$(6) \quad \text{Hom}_{\mathbb{C}[G]}(C^G, B) \cong \text{Hom}_{\mathbb{C}[H]}(C, B|_{\mathbb{C}[H]}).$$

Or, letting  $\chi, \chi^G$ , and  $\phi$  be the corresponding characters, we have:

**Corollary 2.**

$$(7) \quad \langle \chi, \phi_H \rangle_H = \langle \chi^G, \phi \rangle_G.$$

Here  $\langle \cdot, \cdot \rangle_G$  denotes the inner product of characters over  $G$ .

## 4. CLASSIFYING HECKE ALGEBRAS FOR LOW GENUS COVERS

The goal of this section is to describe a simple means of determining whether two Hecke algebras are isomorphic, and an application of this method to classifying the Hecke algebras associated to the Galois groups for low genus branched covering spaces of the sphere.

We first use the following description of Hecke algebras:

**Proposition 3.** Let  $\epsilon_H = \frac{1}{|H|} \sum_{h \in H} h$ . Then  $\mathbb{C}[H \backslash G / H] \cong \text{Hom}_{\mathbb{C}[G]}(1_H^G, 1_H^G)$ , where  $1_H^G$  is the induced representation of the trivial representation on  $H$ .

**Proof:** We have by Lemma 3.19 in [5] that  $\mathbb{C}[H \backslash G / H] \cong \text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G]\epsilon_H, \mathbb{C}[G]\epsilon_H)$ , where  $\mathbb{C}[H \backslash G / H]$  acts on itself via right multiplication. Since  $\epsilon_H$  is the projection of  $\mathbb{C}[H]$  onto the trivial representation  $1_H$ ,  $\epsilon_H \mathbb{C}[H] \cong 1_H$ . Then

$$\begin{aligned} 1_H^G &\cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} \epsilon_H \mathbb{C}[H] \\ &\cong \mathbb{C}[G]\epsilon_H \otimes_{\mathbb{C}[H]} \mathbb{C}[H] \\ &\cong \mathbb{C}[G]\epsilon_H \end{aligned}$$

where the last equality is a basis property of tensor products. ■

The above equality gives a simple way to classify Hecke algebras up to isomorphism. Recall from section II that Hecke algebras are semi-simple, so that they can be decomposed into direct sums of simple algebras, where the decomposition is unique up to isomorphism. Therefore to determine the isomorphism structure of a Hecke algebras, it suffices to know the number and type of these simple algebras. Such data is given by the following:

**Proposition 4.**

$$(8) \quad \text{Hom}_{\mathbb{C}[G]}(1_H^G, 1_H^G) \cong \bigoplus_{i=1}^m M_{n_i \times n_i}$$

where  $m$  is the number of distinct  $G$ -irreducible representations appearing in  $1_H^G$ , and  $n_i$  is the multiplicity of the  $i$ 'th irreducible representation in  $1_H^G$ .  $M_{n_i \times n_i}$  is the algebra of  $n_i \times n_i$  matrices over  $\mathbb{C}$ .

**Proof:** This is just a simple consequence of Schur's lemma:

$$\text{Hom}_{\mathbb{C}[G]}(\psi_i, \psi_j) \cong \begin{cases} \mathbb{C} & i = j \\ 0 & i \neq j \end{cases}$$

for  $\psi_i, \psi_j$   $G$ -irreducibles. Write

$$1_H^G = \bigoplus_{i=1}^m \left( \bigoplus_{j=1}^{n_i} \psi_i \right)$$

with  $\{\psi_i\}$  the  $G$ -irreducible representations. Then using the fact that the  $\text{Hom}(-,-)$  functor commutes with direct sums in both the first and second variables, and applying Schur's lemma, we get the result. ■

Since for any  $G$ -character  $\psi$  and irreducible  $G$ -character  $\chi$ ,  $\langle \psi, \chi \rangle$  is the multiplicity of  $\chi$  in  $\psi$ , we have:

**Corollary 3.** The Hecke algebra  $\mathbb{C}[H \backslash G / H]$  is classified up to isomorphism by the data  $\{\langle 1_H^G, \chi \rangle\}$ . That is,  $\mathbb{C}[H \backslash G / H]$  splits up into a direct sum of  $\mathbb{C}$ -matrix algebras, one for each irreducible  $G$ -character  $\chi$  which appears in  $1_H^G$ , whose dimension is equal to  $\langle 1_H^G, \chi \rangle$ .

By Frobenius Reciprocity, we may recast the above inner product as

$$\langle 1_H^G, \chi \rangle_G = \langle 1_H, \chi_H \rangle_H$$

and so we have an easy way of decomposing  $1_H^G$ .

As an illustration, we list the number of isomorphism classes of Hecke algebras associated to Galois groups  $\text{Gal}(Y/X)$ , where  $Y/X$  is a branched covering of the sphere of genus 2 or 3, distinct up to topological equivalence. This data comes from [1], and was processed using the MAGMA script `decomp_1`. which may be found at the script archive [4].

Dimension	Genus 2	Genus 3
2	3	5
3	4	5
4	6	5
5	2	2
6	2	8
7	1	3
8	3	7
9	0	1
10	1	1
11	0	0
12	4	6
13	1	0
14	0	2
15	0	1
16	1	11
20	0	1
21	0	1
24	3	4
32	0	2
44	0	1
48	1	2
96	0	1
168	0	1

### 5. THE GENERALIZED HOMOLOGY TRACE FORMULA

This section gives a generalization of the homology trace formula for branched covers of the sphere to the case of non-Galois covers.

For the definitions and basic properties of homology groups, see [3],[9],or [13].

Suppose we have a Galois branched cover  $Y/\mathbb{P}^1$  of the sphere,  $n$ -sheeted, with branch points  $Q_1, \dots, Q_t$ . Without loss of generality we may assume the branch points all lie along

the equator of the sphere, since we simply wish to connect the vertices by edges so that they divide the sphere into two faces. For a finite number of vertices this can always be done. So connecting consecutive branch points along the equator by edges  $E_1, \dots, E_t$ , we cover the sphere with two  $t$ -gons  $F_1$  and  $F_2$ , one on the upper hemisphere, the other on the lower hemisphere. Since  $Y$  covers all points of  $\mathbb{P}^1$  evenly save the branch points, each edge  $E_i$  and face  $F_i$  lifts to  $n$  distinct edges (faces, respectively) in  $Y$ , where  $n$  is the degree of the cover. The lifted edges/faces correspond to an entire  $G$ -orbit of edges/faces in  $Y$ , where  $G = \text{Gal}(Y/\mathbb{P}^1)$ . We have fewer vertices lying above the branch points because of ramification. Each branch point  $P_i$  possesses some non-trivial cyclic stabilizer  $\langle c_i \rangle \subset \text{Gal}(Y/\mathbb{P}^1)$ . Hence the vertices lying above  $P_i$  correspond to the cosets  $G/\langle c_i \rangle$ , since  $G$  acts transitively on the vertices (Orbit-Stabilizer Theorem). Let us call such a tiling on  $\mathbb{P}^1$  or the tiling on  $Y$  or constructed from a branched cover an ‘‘equatorial tiling’’. Not that the cover need not be Galois to construct an equatorial tiling.

We may now consider the action of  $G$  on  $H_2(Y)$ ,  $H_1(Y)$ , and  $H_0(Y)$ , the homology spaces spanned by the faces, edges, and vertices, respectively, of the above tiling.  $G$  acts as a group of covering transformations, so it takes faces to faces, edges to edges, etc. This action yields *homology representations* of  $G$ , which can be used to distinguish distinct group actions of  $G$  on various branched covers. In particular, we are interested in the homology representation of  $G$  on  $H_1(Y)$ . In fact, we have the following formula for  $\chi_{H_1(Y)}$ :

$$(9) \quad \chi_{H_1(S)} = 2\rho_0 + (t-2)\rho + \sum_{i=1}^t \rho_i$$

where  $\rho$  is the regular representation of  $G$ ,  $\rho_0$  the trivial representation, and  $\rho_i$  the induced representation of the trivial representation on  $\langle c_i \rangle$ , with  $\langle c_i \rangle$  defined above. A proof of this result can be found in [2].

We shall give a generalization of this formula to the case when  $Y/X$  is not Galois. To do so, we shall trace through the proof of the above formula, making modifications to it to include the case of non-Galois covers.

We shall first need to recall the notion of an Euler-Poincaré map. The following exposition comes from [11]. Suppose we have a map  $\phi$  from the category of  $R$ -modules to an abelian group  $\Gamma$ , such that  $\phi(0) = 0$ , and such that for any exact sequence of  $R$ -modules  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ , we have  $\phi(M) = \phi(L) + \phi(N)$ . Then  $\phi$  is called an *Euler-Poincaré map*.

One example of an Euler-Poincaré map is the map  $\phi$  which counts the dimension of a finite dimensional vector space. If we have a surjective linear transformation  $T$  from  $V$  to  $W$ ,

$$0 \rightarrow \ker T \rightarrow V \rightarrow W \rightarrow 0$$

is exact, and  $\phi(V) = \dim(V) = \phi(\ker T) + \phi(W) = \dim(\ker T) + \dim(\text{Im } T)$  by linear algebra.

We now consider a complex  $E$  of  $R$ -modules such that almost all  $H_i = 0$ . Then one can easily prove the following, based on the definition of  $\phi$ :

**Proposition 5.** Let  $E = \{E_i\}$  be a complex defined as above. Then

$$\sum_i (-1)^i \phi(H_i(E)) = \sum_i (-1)^i \phi(E_i)$$

This proposition allows us to transfer our knowledge of the values of  $\phi$  on the complex  $E$  to its values on the associated homology groups, or vice versa. For instance, if we are given a triangulation, or more generally a tiling by simply-connected polygons, of a connected surface  $S$ , we can let  $C_2$  be the  $\mathbb{C}$ -vector space of 2-chains spanned by the faces of this triangulation,  $C_1$  the space spanned by the edges, and  $C_0$  the space spanned by the vertices. By algebraic topology, we get a complex  $E$  of chain spaces  $\{C_i\}$  associated to  $S$ ,

$$\rightarrow 0 \xrightarrow{\partial} \cdots 0 \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0$$

with  $\partial$  the so-called *boundary operator*. Then again by algebraic topology, when we consider the associated homology groups, we have  $H_i = 0$ ,  $i \geq 3$ ,  $H_2 = H_0 \cong \mathbb{C}$ , and  $H_1 \cong \mathbb{C}^{2\sigma}$ , where  $\sigma$  is the genus of  $S$ . Then applying the map  $\phi$  which counts the dimension of the spaces  $C_i$ ,  $H_i$ , we get

$$\begin{aligned} \sum_i (-1)^i \phi(H_i) &= 1 - 2\sigma + 1 \\ &= \sum_i (-1)^i \phi(C_i) \\ &= \phi(C_2) - \phi(C_1) + \phi(C_0) \\ &= F - E + V \end{aligned}$$

which is the familiar identity of Euler.

In our case, we have a tiling of the  $S$  sphere given by an  $n$ -sheeted branched cover  $S/\mathbb{P}^1$ . We let  $E$  be the complex of chain spaces associated to this tiling, so that  $C_i = 0$ ,  $i \geq 3$ ,  $C_2$  is the space spanned by the faces of the tiling,  $C_1$  the space spanned by the edges, and  $C_0$  the space spanned by the vertices.

Since  $S/\mathbb{P}^1$  is not in general Galois, we must first construct a Galois cover of  $S$  which is Galois over  $\mathbb{P}^1$ . To this end, let  $B$  denote the set of branch points of  $S/\mathbb{P}^1$  and then take the cover  $Y^\circ \rightarrow \mathbb{P}^1 - B$  corresponding to the subgroup  $\Delta(S^\circ, s_0)$ , where  $\Delta(S^\circ, s_0)$  is the *core* of  $\pi_1(S^\circ, s_0)$  in  $\pi_1(\mathbb{P}^1 - B, x_0)$ , defined by

$$\Delta(S^\circ, s_0) = \bigcap_{g \in \pi_1(\mathbb{P}^1 - B, x_0)} g \pi_1(S^\circ, s_0) g^{-1}.$$

One can easily check that this is the largest normal subgroup of  $\pi_1(\mathbb{P}^1 - B, x_0)$  contained in  $\pi_1(S^\circ, s_0)$ , and that  $\Delta(S^\circ, s_0)$  is of finite index in  $\pi_1(\mathbb{P}^1 - B, x_0)$ . Hence by Galois correspondence the closure  $Y/\mathbb{P}^1$  of the associated cover  $Y^\circ/(\mathbb{P}^1 - B)$  is the Galois cover of  $S/\mathbb{P}^1$ .

Then letting  $G = \text{Gal}(Y/\mathbb{P}^1)$ ,  $H = \text{Gal}(Y/S)$ , we can consider the action of the Hecke algebra  $\mathbb{C}[H \backslash G / H]$  on the chain spaces  $C_i(S)$ , along with their associated homology groups  $H_i(S)$ , as follows: First, define the equatorial tilings on  $S$  and then  $Y$  by an iterated lifting of the base equatorial tilings on  $\mathbb{P}^1$ . For any  $C_i(S)$ , we have the natural projection  $p_*$  of  $C_i(Y)$  onto  $C_i(S)$  via the projection  $Y \xrightarrow{p} S$ . So, for any  $\gamma \in C_i(S)$ , let  $\epsilon_H g \epsilon_H$  act on  $\gamma$  by

$$\epsilon_H g \epsilon_H \circ \gamma = p_*(\epsilon_H g \epsilon_H \tilde{\gamma})$$

where  $\tilde{\gamma}$  is any lift of  $\gamma$ . This is well-defined since  $p_*(\tilde{\gamma}) = p_*(\tilde{\delta})$  if and only if  $\epsilon_H \tilde{\gamma} = \epsilon_H \tilde{\delta}$ . To see this, suppose that  $\epsilon_H \tilde{\gamma} = \epsilon_H \tilde{\delta}$ . Then,

$$p_*(\epsilon_H \tilde{\gamma}) = \frac{1}{|H|} \sum_{h \in H} p_*(h \tilde{\gamma}) = \frac{1}{|H|} \sum_{h \in H} p_*(\tilde{\gamma}) = p_*(\tilde{\gamma}),$$

by the  $H$ -invariance of  $p_*$ . It follows that  $p_*(\epsilon_H \tilde{\gamma}) = p_*(\epsilon_H \tilde{\delta})$ . Now suppose that  $p_*(\tilde{\gamma}) = p_*(\tilde{\delta})$ . We may write  $\epsilon_H \tilde{\gamma}$  uniquely as

$$(10) \quad \epsilon_H \tilde{\gamma} = a_1 \epsilon_H \tilde{\beta}_1 + \cdots + a_s \epsilon_H \tilde{\beta}_s,$$

where the  $H\tilde{\beta}_1, \dots, H\tilde{\beta}_s$  are the  $H$ -orbits of the  $i$ -cells of  $Y$ . It follows that

$$(11) \quad p_*(\epsilon_H \tilde{\gamma}) = a_1 p_*(\tilde{\beta}_1) + \cdots + a_s p_*(\tilde{\beta}_s).$$

If  $p_*(\tilde{\gamma}) = p_*(\tilde{\delta})$ , then it follows that  $\epsilon_H \tilde{\delta}$  has the same expansion as  $\epsilon_H \tilde{\gamma}$  in (11), since  $\{p_*(\tilde{\beta}_1), \dots, p_*(\tilde{\beta}_s)\}$  is a basis for  $C_i(S)$ . Note that the above argument shows that  $p_* : \epsilon_H C_i(Y) \rightarrow C_i(S)$  is an isomorphism.

In the case of  $C_2(S)$  or  $C_1(S)$ , we may reinterpret this action as follows: since  $Y/\mathbb{P}^1$  is Galois,  $G$  permutes simply transitively the lifts of all the faces (or edges) in  $\mathbb{P}^1$ . Hence we may relabel all these lifts by their corresponding group elements in  $G$ . Then the projection map  $p$  amounts to just left multiplying by the idempotent  $\epsilon_H$ , and the action of  $\mathbb{C}[H \backslash G / H]$  on  $C_i(S)$  for  $i = 1, 2$  is just the action of  $\mathbb{C}[H \backslash G / H]$  on the right  $H$  cosets  $\epsilon_H g$ . Since these right cosets form the  $\mathbb{C}[G]$ -module  $\epsilon_H \mathbb{C}[G]$ ,  $\mathbb{C}[H \backslash G / H]$  acts on  $\epsilon_H \mathbb{C}[G]$  by left multiplication. Then  $\mathbb{C}[H \backslash G / H]$  simply acts on the left as the algebra of  $\mathbb{C}[G]$  endomorphisms of  $\epsilon_H \mathbb{C}[G]$ . From the discussion in Proposition 3, this action is equivalent to the action of  $\mathbb{C}[H \backslash G / H]$  on itself via left multiplication.

Note that for a Galois cover  $S/\mathbb{P}^1$  this left regular representation reduces to just the left regular representation of the Galois group  $G/H$ .

Now noting that as  $\mathbb{C}[H \backslash G / H]$ -modules,  $C_2(S)$  decomposes into a direct sum of two isomorphic subspaces (one for the upper tile, the other for the lower), and  $C_1(S)$  decomposes into a direct sum of  $t$  isomorphic subspaces (one for each edge), then along with Corollary 1, we have

**Proposition 6.** The homology representations of  $\mathbb{C}[H \backslash G / H]$  on  $C_2(S)$  and  $C_1(S)$  are given by

- $\chi_{C_2(S)} = \tilde{\rho} \oplus \tilde{\rho}$
- $\chi_{C_1(S)} = \bigoplus_{i=1}^t \tilde{\rho}$

where  $\tilde{\rho}$  is the left regular representation of  $\mathbb{C}[H \backslash G / H]$  on itself.

The homology representation of  $\mathbb{C}[H \backslash G / H]$  on  $C_0(S)$  is a bit trickier. Here,  $G$  does not act simply transitively on the vertices of the tiling, since the stabilizers of the branch points  $\{P_i\}$  are non-trivial cyclic subgroups  $\langle c_i \rangle$ . So instead of substituting any element of  $G$  for  $\tilde{\gamma}$ , we must substitute left cosets  $g \langle c_i \rangle$ , for fixed  $c_i$ . Then we get an action of  $\mathbb{C}[H \backslash G / H]$  on the double cosets  $\epsilon_H g \epsilon_{\langle c_i \rangle}$ .

To reinterpret this action, we need the following result from representation theory

**Theorem 1** (Mackey). Let  $H, K$  be subgroups of  $G$ , and  $W$  a  $\mathbb{C}[H]$ -representation. Then

$$(12) \quad (W^G)_K = \bigoplus_{t \in T} (W_{H^t \cap K})^K,$$

where  $T$  is any set of  $(H, K)$  double coset representatives, and  $H^t = tHt^{-1}$ .

In our case, with  $\mathcal{H} = \mathbb{C}[H \backslash G / H]$ , we are interested in the  $\mathcal{H}$ -representation on  $\epsilon_H \mathbb{C}[G] \epsilon_{\langle c_i \rangle}$ . Or, since  $\epsilon_H \mathbb{C}[H] = \mathbb{C}[H] \epsilon_H = \epsilon_H$ , this representation is equivalent to  $\mathcal{H} \otimes_{\mathbb{C}[H]} \mathbb{C}[G] \epsilon_{\langle c_i \rangle}$ .

Since  $\mathcal{H}$  can be considered as a  $(\mathcal{H}, H)$  bi-module, for any irreducible  $\mathcal{H}$ -representation  $\phi$  we can apply the change of rings formula to get:

$$\text{Hom}_{\mathcal{H}}(\mathcal{H} \otimes_{\mathbb{C}[H]} \mathbb{C}[G] \epsilon_{\langle c_i \rangle}, \phi) = \text{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G] \epsilon_{\langle c_i \rangle}, \phi|_{\mathbb{C}[H]}).$$

Since as a  $\mathbb{C}[H]$ -module,  $\mathbb{C}[G]\epsilon_{\langle c_i \rangle} \cong ((1_{\langle c_i \rangle})^G)_H$  we can apply Frobenius Reciprocity to get:

$$\langle \chi_{\mathcal{H} \otimes_{\mathbb{C}[H]} \mathbb{C}[G]\epsilon_{\langle c_i \rangle}}, \phi \rangle = \langle ((1_{\langle c_i \rangle})^G)_H, \phi_H \rangle$$

Applying Mackey's Theorem to the last expression yields:

**Proposition 7.** Let  $c_i, H, G$ , and  $\phi$  be as above,  $T$  any set of  $(H, \langle c_i \rangle)$ -double coset representatives. Then

$$\begin{aligned} \langle ((1_{\langle c_i \rangle})^G)_H, \phi_H \rangle &= \sum_{t \in T} \langle (1_{\langle c_i \rangle})_{\langle c_i \rangle^t \cap H}, \phi_H \rangle \\ &= \sum_{t \in T} \langle (1_{\langle c_i \rangle^t \cap H})^H, \phi_H \rangle \\ &= \sum_{t \in T} \langle (1_{\langle c_i \rangle^t \cap H}), \phi_{\langle c_i \rangle^t \cap H} \rangle_{\langle c_i \rangle^t \cap H} \end{aligned}$$

The last two lines are by transitivity of restriction and Frobenius Reciprocity.

This characterizes the action of  $\mathcal{H}$  on the ramification points over a single branch point  $P_i$ . Then the representation of  $\mathcal{H}$  on  $C_0(S)$  breaks up into a direct sum of  $t$  subrepresentations, each acting on the fiber of some  $P_i$  in  $Y$  (or equivalently,  $\mathcal{H}$  acts on  $\epsilon_H \mathbb{C}[G]\epsilon_{\langle c_i \rangle}$ ,  $i = 1, \dots, t$ ). Since for compact connected Riemann surfaces,  $H_2(S) \cong H_0(S) \cong \mathbb{C}$ , so  $\mathcal{H}$  acts trivially on these spaces. Then using the fact that the trace of a Hecke algebra element is an Euler-Poincaré map, we get

$$(\tilde{\rho} + \tilde{\rho}) - \left( \sum_{i=1}^t \tilde{\rho} \right) + \left( \sum_{i=1}^t \tilde{\rho}_i \right) = \tilde{\rho}_0 - \chi_1(S) + \tilde{\rho}_0$$

where  $\tilde{\rho}_0$  is the trivial representation of  $\mathcal{H}$  on itself,  $\tilde{\rho}$  is the regular representation, and  $\tilde{\rho}_i$  is the action of  $\mathcal{H}$  on  $\epsilon_H \mathbb{C}[G]\epsilon_{\langle c_i \rangle}$ .

Expressing the above characters in terms of  $G$ -characters, we get

**Proposition 8.** The homology representation of the Hecke algebra  $\mathbb{C}[H \backslash G / H]$  decomposes as

$$(13) \quad \chi_1(S) = 2\rho_0 + (t-2) \sum_{\chi \in X(G)} \langle \chi, 1_H^G \rangle \chi + \sum_{i=1}^t \left[ \sum_{\phi \in X(G)} \left( \sum_{a \in T_i} \langle (1_{\langle c_i \rangle^a \cap H}), \phi_{\langle c_i \rangle^a \cap H} \rangle \phi \right) \right]$$

where  $T_i$  is any set of  $(H, \langle c_i \rangle)$ -double coset representatives, and  $X(G)$  is the set of irreducible  $G$ -characters.

This establishes the general homology formula. Note that for Galois covers,  $H = G$ , and we get back equation (9).

## 6. QUESTIONS

There is one immediate question which presents itself for investigation. It regards bounding the dimension of a Hecke algebra  $\mathbb{C}[H \backslash G / H]$  in terms of the genus of the covering space and some independent constant. In the case of Galois covers, we have the familiar *Hurwitz bound* on the size of the Galois group, given by

$$(14) \quad |G| \leq 84(\sigma - 1)$$

for surfaces with  $\sigma \geq 1$ .

For non-Galois covers, the dimension of  $\mathbb{C}[H \backslash G / H]$  is just the number of  $(H, H)$  double cosets in  $G$ . As Ellenberg remarks in [7], it would be interesting to see if a similar such bound existed for general branched covers, i.e., if there were a constant  $\gamma$  independent of  $G$  and  $H$  such that

$$(15) \quad \dim(\mathbb{C}[H \backslash G / H]) \leq \gamma(\sigma - 1)$$

for surfaces with  $\sigma \geq 1$ .

$\dim(\mathbb{C}[H \backslash G / H]) \leq \gamma(\sigma - 1)$ . We note that such a question could be amenable to computational exploration using a program such as MAGMA.

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