Oval Intersections in Tilings on Surfaces

Dennis A. Schmidt[¤]

November 26, 1998

Abstract

A tiling is a covering by polygons, without gaps or overlapping, of a compact, orientable surface. We are particularly interested in tilings by triangles that generate a large symmetry group of the surface. An oval of the tiling is a simple, closed curve that is a union of edges of the tiling. We investigate the number of points of intersection of two ovals. We have found that the number of intersections is bounded when the subgroup of orientation preserving symmetries is abelian. However, there is no upper bound on the number of intersections in the non-abelian case.

1 Introduction

Let S be a compact, orientable surface of genus ¾. Suppose that S has been covered by a pattern of polygons, as in the pattern of triangles on the sphere in Figure 1 and the torus in Figure 2. For this paper, a tiling is a covering by congruent triangles, called tiles, without gaps or overlapping, of a compact, orientable surface.

Select an edge e on a tile from Figure 1 or Figure 2. Now e continues through its endpoint to connect with a unique edge e^0, belonging to another tile, to form $C = e$ [e^0; a smooth, simple curve (no angle where e and e^0 join). If we continue to adjoin edges to C in this manner we obtain a union of edges which is a smooth simple, closed curve. Let us call such a curve an <u>oval</u> and denote it by O_e since it is determined by e:

[¤]Author supported by NSF-REU grant # DMS 9322338

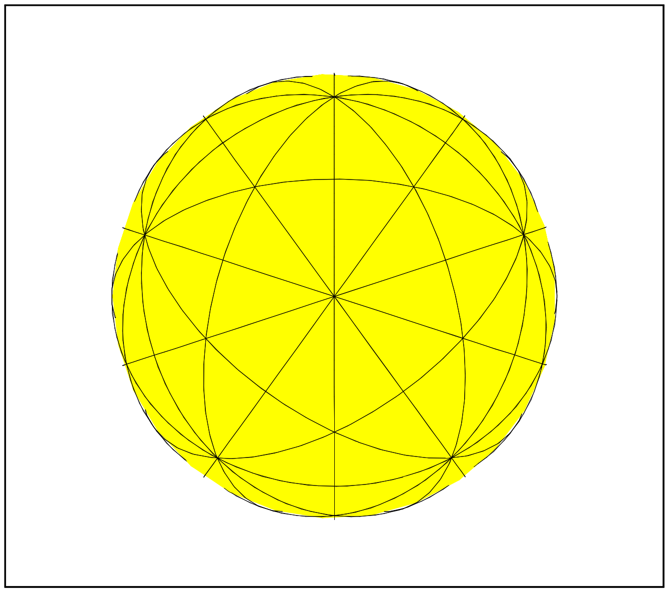

Figure 1. Tiling of Triangles on a Sphere

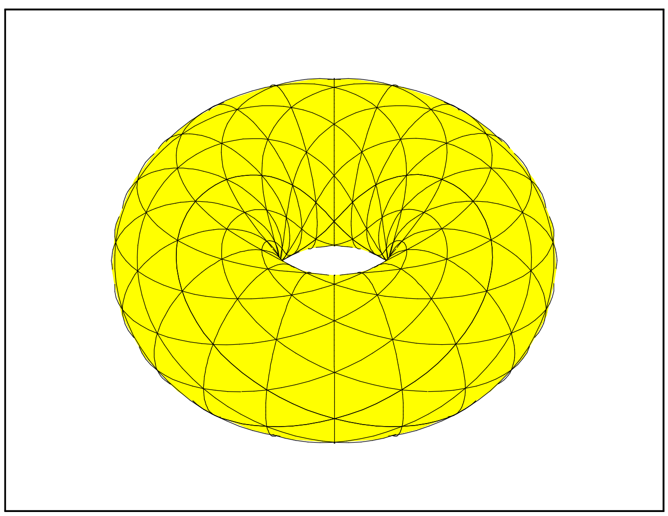

Figure 2. Tiling of Triangles on a Torus

Notice in Figure 1, that the tiles joined along an edge are mirror re‡ections of each other across the edge. The same is true for some of the tiles in Figure 2 and would be true for all tiles if the surface were a metric such that it has constant curvature. We shall assume that all our surfaces have metrics giving them constant curvature. Recall that for such surfaces, a re‡ection r_e in an edge e of a tile is an orientation preserving isometry of the surface S that fixes the edge e. The mirror of the reflection is $\{x \in S : r_e(x) = x\}$ which is a finite set of geodesic curves containing e. Each of the closed geodesic curves in the mirror is called an oval. We put the above observations into a definition.

Definition 1 A tiling is called a geodesic, kaleidoscopic tiling if it meets the following conditions.

- Two tiles meeting each other in an edge are mirror images of each other in the edge, i.e., each edge e in the tiling determines a reflection r_e of the entire surface which fixes e pointwise (kaleidoscopic property).
- The mirror, $\{x \in S : r_e(x) = x\}$, of each such reflection r_e is a union of edges of the tiling (geodesic property).

Reflections preserve geometric structure, such as geodesics, distances, angles, and area. Notice that in the tiling on the sphere, the ovals always intersect at exactly two vertices. The tiling on the torus exhibits instances where two ovals only intersect at one vertex and some at two vertices. This paper investigates the intersections of two ovals in triangular tilings on surfaces of genus $\sigma \geq 2$. We will prove results for both abelian and non-abelian tiling groups, along with an investigation of oval intersection patterns.

Acknowledgments The author would like to thank Barry Balof (Colorado College), S. Allen Broughton (Rose-Hulman Institute of Technology), and Eric Farmer (Kansas State University) for useful conversations and suggestions during the NSF-REU program at Rose-Hulman where the research was carried out. Group theoretic experimentation and computations were done using the MAGMA software package [1].

2 The Tiling Group of S

If we have a kaleidoscopic tiling, we select a tile T_0, called the master tile (Figure 3), and denote the sides of our master tile by $p, q,$ and r which also denote the corresponding reflections in the sides of T_0. The vertices opposite $p, q,$ and r we label $P, Q,$ and R, respectively as in Figure 3. Since each of $p, q,$ and r is an isometry, the reflections generate a group of isometries $G^* = \langle p, q, r \rangle$, called the tiling group of S.

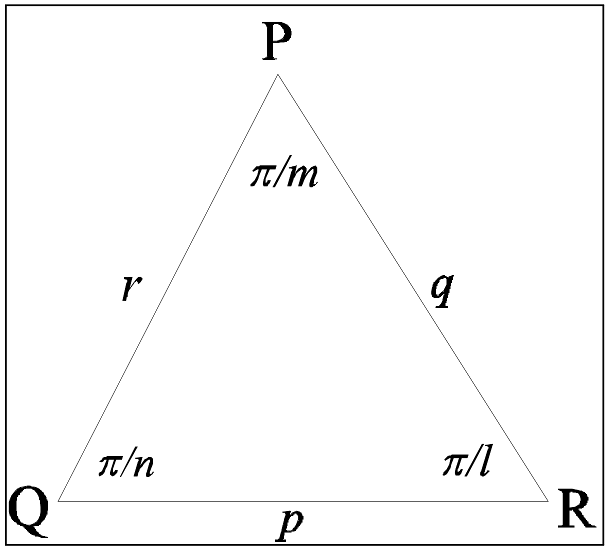

Figure 3. Master Tile with Generators of G^*

Remark 1 There may be isometries of S that preserve the tiling but are not contained in G^*: This will only happen if there are symmetries of S that map the master tile back to itself. Since the term "symmetry group" of an object usually refers to the totality of self-transformations of the object preserving the given structure; the use of the term symmetry group to describe G^* would be misleading in this case. Therefore, we use the term tiling group.

The product $a = pq$, which fixes the vertex R, is a counter-clockwise rotation through $\frac{2\pi}{l}$ radians, where l is half the number of triangles surrounding R. Similarly $b = qr$ and $c = rp$, which fix P and Q respectively, are counter-clockwise rotations through $\frac{2\pi}{m}$ and $\frac{2\pi}{n}$ radians. This triangle will have angles $\frac{\pi}{l}; \frac{\pi}{m}; \frac{\pi}{n}$ (for some integers $l; m; n$). We call such a triangle an $(l; m; n)$-triangle. It follows that the order of the elements $a; b; c$ are exactly of order $l; m; n$ respectively:

$$o(a) = l; \; o(b) = m; \; o(c) = n: \tag{1}$$

Since $p; q;$ and r are reflections then $p^2 = q^2 = r^2 = 1$ and hence $abc = pqqrrp = 1$; leading to our second basic relation:

$$abc = 1: \tag{2}$$

The conjugation action of q on a and b induces an automorphism μ which maps the generators a and b to their inverses:

$$\mu(a) = a^{-1}; \; \mu(b) = b^{-1}: \tag{3}$$

To see this we observe that:

$$\begin{aligned} \theta(a) &= q^{-1}aq = q(pq)q = qp = q^{-1}p^{-1} = a^{-1} \text{ and} \\ \theta(a) &= q^{-1}bq = q(qr)q = rq = r^{-1}q^{-1} = b^{-1}. \end{aligned}$$

Since q is a reflection, θ is an involutary automorphism, i.e., $\theta^2 = 1$.

Let G be the subgroup of orientation preserving elements G^*, we will call G the (orientation preserving) tiling group or OP tiling group generated by the tiling. It is easily proven that G is of index 2 in G^*, that $G = \langle a, b, c\rangle = \langle a, b\rangle$, and $G^* \cong \langle\theta\rangle \ltimes G$. The relationship between the order of G and the genus σ is given by the Riemann-Hurwitz equation [2]:

$$\frac{2\sigma - 2}{|G|} = 1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right). \tag{4}$$

If G is an arbitrary finite group and a triple (a, b, c) from G generates all of G and satisfies (1) and (2), we call it an generating $(l, m, n)-$triple of G. Note that the automorphism θ is unique if it exists since $G = \langle a, b\rangle$. It turns out that the conditions (1) - (4) above are also sufficient to generate a tiling as follows.

Proposition 1 Let (a, b, c) be a generating (l, m, n)-triple of the finite group G, assume there exists an automorphism θ of G which satisfies (3), and that number σ defined by (4) is an integer. Then, there is a surface S with a geodesic, kaleidoscopic tiling by (l, m, n)-triangles and an action of $G^* = \langle\theta\rangle \ltimes G$ on S such that a, b, c and $p = a\theta$, $q = \theta$ and $r = \theta c$ are the elements derived from the master tile as above.

The proof is easily constructed from results in [5]. As a consequence we immediately get.

Proposition 2 Every generating (l, m, n)-triple for an abelian OP tiling group defines a tiling on a surface, provided that σ, defined by the equation (4), is an integer.

Proof. Let (a, b, c) be a generating (l, m, n)-triple for G. Consider the mapping $\theta(x) = x^{-1}$ on G. It is elementary to show that θ is an involutary automorphism satisfying (3). ■

Example 1 Let $G = Z_{12} = \langle x\rangle$. Setting $(a, b, c) = (x^8, x^3, x)$ and $\theta(x) = x^{-1}$ we obtain a surface of $\sigma = 3$ with a tiling by $(3, 4, 12)$-triangles.

2.1 Parity, Edges, Vertices, Ovals, & Transitivity

We will need consider eight different types of triangles depending on the parity of l, m, n. For example, if l is odd, m is even, and n is even, we will label this as the OEE case. The eight cases are: OOO, EEO, EOE, OEE, OOE, OEO, EOO, and EEE. We choose to reduce the number of cases we consider to OOO, EOE, OOE, and EEE because any other type of mixed case can be derived by relabeling the sides of the triangles. The analysis breaks into cases because any two reflections in edges at an odd vertex are conjugate, though this need not hold at even vertices. Furthermore, the pattern of edges on an oval is determined by the parity pattern of (l, m, n). Define the integers λ, μ, ν, depending on the parity of (l, m, n), by the following table.

Table 1

Order	odd	even
l	$l = 2\lambda + 1$	$l = 2\lambda$
m	$m = 2\mu + 1$	$m = 2\mu$
n	$n = 2\nu + 1$	$n = 2\nu$

In our calculations and derivations to follow we will be making much use of the group action and labeling the vertices, edges and tiles by group elements. Thus, we now discuss the orbits and stabilizers of G and G^* on the geometry of S. The group G^* acts simply transitively on the triangles of the tiling, thus for each tile T there exists a unique $t \in G^*$ that maps the master tile to T; i.e., tile $T = tT_0$. Every vertex in the tiling is equivalent to exactly one vertex on the master tile, because of simple transitivity. Hence we call a vertex an R-type vertex if it is equivalent to R and similarly for P, Q. The element $a \in G$ fixes the vertex R, similarly b fixes P and c fixes Q. In fact,

$$\begin{aligned} G_R &= \mathrm{stab}_G(R) = \langle a \rangle, \\ G_P &= \mathrm{stab}_G(P) = \langle b \rangle, \\ G_Q &= \mathrm{stab}_G(Q) = \langle c \rangle. \end{aligned}$$

An edge e of a tile is G-equivalent to a unique edge of the master tile, again by simple transitivity. We call an edge e a p, q or r-type edge accordingly. The G-stabilizers of edges are trivial.

Ovals are a union of edges in the tiling. If we consider an oval O, such that $O \cap T_0 = e$ corresponds to one of the sides of T_0, then we denote the oval as one of $O_p, O_q,$ or O_r, as appropriate. Since every edge is equivalent to an edge of the master tile then every oval in the tiling is equivalent to an oval bordering the master tile, i.e., one of $O_p, O_q,$ or O_r. Ovals can only intersect at vertices, but there are several tiles around a single vertex. We need to find a convention so that every tile surrounding a vertex V will label V the same. Consider a tile $T = tT_0$ ($t \in G$) and one of its vertices V of an arbitrary type $\tilde{V} \in \{P, Q, R\}$. If $\mathrm{stab}_G(\tilde{V}) = \langle v \rangle$, then the tiles containing V, associated with elements in G, have the form $tv^i T_0$. We label V as the coset $t\langle v \rangle$.

Since two ovals can only intersect at vertices of the tiling, it is convenient to denote an oval by a set of vertices, $O = \{V_1, V_2, \ldots, V_w\}$, derived from the endpoints of edges making up the oval (even though the vertices do not uniquely determine the oval). We define the length of an oval to be the number of vertices in the oval, which equals the number of edges. For notational purposes, we denote the length of the oval as $\|O\| = \#\{V_1, V_2, \ldots, V_w\}$. The number of intersections between two ovals is $\#(O_x \cap O_y)$.

2.2 Oval Patterns and Lengths

The group of cyclic rotations of an oval coming from G is called the rotational group of O, denoted $\mathrm{Rot}(O)$. In Table 2 below for each of the ovals O_p, O_q and O_r we have given a generator for $\mathrm{Rot}(O)$, the oval's vertex pattern, and the oval length in the four different parity patterns OOO, OOE, EOE, and EEE. We have denoted the generators of the rotational groups of the ovals O_p, O_q and O_r by $h_p, h_q,$ and h_r respectively. The vertex pattern describes the cyclic pattern of vertices encountered in an oval as we move along the oval in the positive direction specified by the edge. Thus for O_p the vertex R is always encountered first, since we consider p to be the oriented segment $\overrightarrow{PR}$. In column 4 of Table 2, the length of an oval is the number of vertices in the vertex pattern multiplied by the order of h ($\mathrm{Rot}(O) = \langle h \rangle$). The elements h were found by applying the general results in [3] and they may also be found in [4].

From Table 2, we can easily derive the vertex labeling scheme shown in detail in Table 3 at the end of the paper. For example, consider the oval O_p in the OOO case. As we move along O_p in the direction of p from R to P we encounter the following vertices R; $a^{\lambda+1}P$; $a^{\lambda+1}b^{\mu+1}Q$; $a^{\lambda+1}b^{\mu+1}c^{\nu+1}R = h_pR$; $a^{\lambda+1}b^{\mu+1}c^{\nu+1}a^{\lambda+1}P = h_p a^{\lambda+1}P; \ldots$; etc. This immediately gives the cosets in

Table 3.

Table 2

Case	Rot(O) = $\langle h \rangle$	Vertex Pattern	$\|O\|$
OOO	$h_p = a^{\alpha+1} b^{\beta+1} c^{\gamma+1}$	RPQ	$3 \cdot o(h_p)$
	$h_q = b^{\beta+1} c^{\gamma+1} a^{\alpha+1}$	PQR	$3 \cdot o(h_q)$
	$h_r = c^{\gamma+1} a^{\alpha+1} b^{\beta+1}$	QRP	$3 \cdot o(h_r)$
OOE	$h_p = a^{\alpha+1} b^{\beta+1} c^{\gamma} b^{\beta} a^{\alpha} c^{\gamma}$	RPQPRQ	$6 \cdot o(h_p)$
	$h_q = b^{\beta+1} c^{\gamma} b^{\beta} a^{\alpha} c^{\gamma} a^{\alpha+1}$	PQPRQR	$6 \cdot o(h_q)$
	$h_r = c^{\gamma} b^{\beta} a^{\alpha} c^{\gamma} a^{\alpha+1} b^{\beta+1}$	QPRQRP	$6 \cdot o(h_r)$
EOE	$h_p = a^{\alpha} c^{\gamma}$	RQ	$2 \cdot o(h_p)$
	$h_q = b^{\beta+1} c^{\gamma} b^{\beta} a^{\alpha}$	PQPR	$4 \cdot o(h_q)$
	$h_r = c^{\gamma} b^{\beta} a^{\alpha} b^{\beta+1}$	QPRP	$4 \cdot o(h_r)$
EEE	$h_p = a^{\alpha} c^{\gamma}$	RQ	$2 \cdot o(h_p)$
	$h_q = b^{\beta} a^{\alpha}$	PR	$2 \cdot o(h_q)$
	$h_r = c^{\gamma} b^{\beta}$	QP	$2 \cdot o(h_r)$

2.3 Quick Facts

Here we note some elementary observations about certain orientation preserving tiling groups and parity patterns.

Lemma 1 Let G be an abelian OP tiling group. In the OOO case, $\|O_p\| = \|O_r\| = \|O_q\|$. In the EOE case, $2 \cdot \|O_p\| = \|O_r\| = \|O_q\|$.

Proof. Our result follows by Table 2, since $a, b,$ and c commute and hence $h_p = h_q = h_r$. ■

Lemma 2 Suppose G is an OP tiling group of odd order. Then G can only produce an OOO tiling.

Proof. A group of odd order cannot have an element of even order. ■

Lemma 3 Suppose the OP tiling group G is the dihedral group $G = D_s = \langle r, f : r^s = f^2 = 1, rf = fr^{s-1} \rangle$ on a regular s-gon. Then G can only produce an EOE, or EEE tiling.

Proof. We may write $G = \langle r \rangle \cup f\langle r \rangle$. The elements in $f\langle r \rangle$ all have order 2 so that odd order elements all belong to the cyclic subgroup $\langle r \rangle$. Since $G = \langle a, b \rangle$ then the OOO and OOE cases are not possible since we get the contradiction $G = \langle a, b \rangle \subseteq \langle r \rangle$. ■

3 Bounds for Oval Intersections

We can determine every type of oval intersection by examining all the possible ovals around the master tile. We note that because an oval is a simple, closed curve, an oval can never intersect itself.

3.1 Abelian Tiling Groups

If we use an abelian tiling group to generate a tiling, we will see that the length of an oval is bounded. We simplify our work by eliminating the possibility of an OOE tiling with an abelian tiling group.

Lemma 4 If $G = \langle a, b, c\rangle$ is an abelian OP tiling group with generating triple (a, b, c) that satisfies (1) and (2), then we cannot have exactly one of a, b, c with even order.

Proof. We may assume that a, b have odd order and that c has even order. Now, $(ab)^{lm} = a^{lm}b^{lm} = 1$, so $o(ab)$ must divide lm. Therefore $o(ab)$ is odd, hence $o(c) = o((ab)^{-1}) = n$ is odd, a contradiction. ■

This lemma allows us to restrict our attention to the EEE, OOO, and EOE cases for abelian tiling groups. Now we show that all oval lengths are bounded for abelian tiling groups.

Lemma 5 If a tiling has an abelian OP tiling group G, then the length of any oval in the tiling is ≤ 8.

Proof. By Lemma 4, we have the following cases: OOO, EOE, and EEE.

Case OOO: By Lemma 1, every oval has length $\|O\| = 3 \cdot o(a^{\lambda+1}b^{\mu+1}c^{\nu+1})$. But $(a^{\lambda+1}b^{\mu+1}c^{\nu+1})^2 = a^{2\lambda+2}b^{2\mu+2}c^{2\nu+2} = (a^{2\lambda+1}b^{2\mu+1}c^{2\nu+1})(abc) = 1$ and so $a^{\lambda+1}b^{\mu+1}c^{\nu+1}$ has order 1 or 2. But $|G|$ is odd, since G is generated by odd order elements. If follows that $a^{\lambda+1}b^{\mu+1}c^{\nu+1} = 1$ and, therefore, $\|O\| = 3$.

Case EOE: Since $(a^{\lambda}c^{\nu})^2 = a^{2\lambda}c^{2\nu} = 1$, it follows from Lemma 1 that $\|O_p\| \leq 4$, and $\|O_q\|, \|O_r\| \leq 8$.

Case EEE: We have already verified $\|O_p\| \leq 4$. Similarly $(a^{\lambda}b^{\mu})^2 = (b^{\mu}c^{\nu})^2 = 1$, and hence $\|O_q\|, \|O_r\| \leq 4$ by Lemma 1. ■

With Lemmas 4 and 5, we can easily derive Theorem 1.

Theorem 1 The number of intersections between two ovals in a tiling with an abelian tiling group is at most 8.

Proof. By Lemma 5, the length of any oval is at most 8, hence containing at most 8 vertices. Therefore, two ovals can only intersect each other at most 8 times. ■

Here is an example that illustrates Theorem 1.

Example 2 Let $G = Z_n$ be a tiling group where $n \geq 3$ is odd, and set $\theta(x) = x^{-1}$. Then, (x, x, x^{n-2}) is a generating (n, n, n) triple ($\gcd(n, n-2) = 1$) for a tiling on a surface of genus $\sigma = \frac{n-1}{2}$. Since $\frac{|G|}{|G_R|} = \frac{|G|}{|G_P|} = \frac{|G|}{|G_Q|} = 1$ there is only one vertex of each type in the tiling. The length of all ovals is 3 and all ovals have the same three vertices. Hence, every oval intersects every other oval in exactly 3 vertices.

3.2 Non-Abelian Tiling Groups

The bound set forth by Theorem 1 need not hold for non-abelian tiling groups. Theorem 2 will construct a sequence of non-abelian tiling groups where the number of intersections between two ovals increases without bound.

Proposition 3 Let $G = Z_t \times D_s = \langle w \rangle \times \langle r, f \rangle$ with $o(w) = t$ and r, f as defined above in Lemma 3. Assume that s and t are relatively prime and $t \geq 3$ is odd. Define

$$a = (1, f),\; b = (w, r),\; c = (w^{-1}, r^{-1}f),\; \text{and}\; \theta(x, y) = (x^{-1}, fyf).$$

Then (a, b, c) is a generating $(2, st, 2t)$-triple, and θ is an involutary automorphism of G.

Proof. It is routine to show that θ is an automorphism of G. We show that θ maps a, b to their inverses and that $abc = 1$.

$$\begin{aligned}
\theta(a) &= \theta((1, f)) = (1^{-1}, fff) = (1, f) = (1, f)^{-1} = a^{-1}, \\
\theta(b) &= \theta((w, r)) = (w^{-1}, frf) = (w^{-1}, r^{-1}) = (w, r)^{-1} = b^{-1}, \\
abc &= (1, f)(w, r)(w^{-1}, r^{-1}f) = (1ww^{-1}, frr^{-1}f) = (1, 1) = 1.
\end{aligned}$$

Now as s and t are relatively prime, b generates a cyclic group of order st containing both $\langle w \rangle \times \langle 1 \rangle$ and $\langle 1 \rangle \times \langle r \rangle$. It follows that a and b generate G. ■

Lemma 6 Let G, a, b, c, and θ be defined above and suppose $t \geq 3$ is odd and $\gcd(s, t) = 1$. Then, G is the OP tiling group for a $(2, st, 2t)$-tiling on a surface with $\sigma = \frac{s(t-1)}{2}$ and oval lengths

$$\|O_p\| = 2s \text{ and } \|O_q\| = \|O_r\| = 4.$$

Proof. By Proposition 2, there exists a $(2, st, 2t)$ tiling on a surface with genus $\sigma = \frac{s(t-1)}{2}$. We compute the length of the ovals.

First note that $o(a) = l = 2$, $o(b) = m = st$, $o(c) = n = 2t$. Obviously l, n are always even. Therefore we have two possible cases: EEE, EOE. From Table 2, we can compute the lengths of the ovals. In the calculations note that $r^j f = f r^{-j}$ has order 2 so that $(r^j f)^t = r^j f$ and $r^{\frac{st}{2}}$ has order 2 when s is even.

Case EEE: (s is even)

$$\begin{aligned}
\|O_p\| &= 2 \cdot o(a^\lambda c^\gamma) = 2 \cdot o((1; f)(w^{-t}; (r^{-1} f)^t) = 2 \cdot o((1; f r^{-1} f)) \\
&= 2 \cdot o((1; r)) = 2s \\
\|O_q\| &= 2 \cdot o(a^\lambda b^\mu) = 2 \cdot o((1; f)(w^{\frac{st}{2}}; r^{\frac{st}{2}}) = 2 \cdot o((w^{\frac{st}{2}}; f r^{\frac{st}{2}})) = \\
&= 2 \cdot o((w^{\frac{(s-1)t}{2}}; f r^{\frac{st}{2}})) = 4 \\
\|O_r\| &= 2 \cdot o(b^\mu c^\gamma) = 2 \cdot o((w^{\frac{st}{2}}; r^{\frac{st}{2}})(w^{-t}; (r^{-1} f)^t)) = 2 \cdot o(1; r^{\frac{st}{2}-1} f)) = 4
\end{aligned}$$

Case EOE: (s is odd)

$$\begin{aligned}
\|O_p\| &= 2s \text{ (same as above).} \\
\|O_q\| &= 4 \cdot o(b^{\mu+1} c^\gamma b^\mu a^\lambda) = 4 \cdot o((w; r)^{\frac{st+1}{2}} (w^{-t}; (r^{-1} f)^t)(w; r)^{\frac{st-1}{2}} (1; f)) \\
&= 4 \cdot o((w^{st-t}; r^{\frac{st+1}{2}} r^{-1} f r^{\frac{st-1}{2}} f)) = 4 \cdot o((1; r^{\frac{st+1}{2}} r^{-1} r^{-\frac{st-1}{2}})) = 4 \\
\|O_r\| &= 4 \cdot o(b^\mu a^\lambda b^{\mu+1} c^\gamma) = 4 \cdot o((w; r)^{\frac{st-1}{2}} (1; f)(w; r)^{\frac{st-1}{2}+1} ((w^{-t}; (r^{-1} f)^t)) \\
&= 4 \cdot o((w^{st-t}; r^{\frac{st-1}{2}} f r^{\frac{st-1}{2}} f)) = 4
\end{aligned}$$

■

Lemma 7 Let G be defined as above and consider $O_1 \neq O_2$ both equivalent to O_p, then $\#(O_1 \cap O_2) = s$.

Proof. Let $V_P(X), V_Q(X)$, and $V_R(X)$ denote the set of P, Q, and R-type vertices in a set X.

Case EEE: By Lemma 6, $\|O_p\| = 2s$. Also by Table 2, O_p is made up of alternating Q and R-type vertices. Hence, $\#(V_Q(O_p)) = \#(V_R(O_p)) = s$. Since $t \geq 3$, there are at least 6 ovals going through any Q-type vertex. These are alternating O_r and O_p-type ovals. Therefore there are at least two intersecting ovals that are equivalent to O_p. Consider two ovals O_p^1 and O_p^2 that are equivalent to O_p. Since $\frac{|G|}{|G_Q|} = \frac{2st}{2t} = s$, then there are exactly s Q-type vertices in the entire tiling, hence $\#(O_p^1 \cap O_p^2) \geq s$. By Table 4 at the end of the paper, we can see that for any O_p^1; and O_p^2 that are equivalent to O_p, the intersection $V_R(O_p^1) \cap V_R(O_p^2) = \emptyset$. Hence $\#(O_p^1 \cap O_p^2) = s$.
Case EOE: This is similar to Case EEE. ■

Theorem 2 There exists a sequence of non-abelian tiling groups where the number of intersections between two ovals increases without bound.

Proof. Suppose $G_s = Z_t \times D_s$ where t is fixed, odd and $\gcd(s, t) = 1$. By Lemmas 5, 6, and 7 we can construct a sequence of non-abelian tiling groups $\{G_s\}$, such that for each G_s, there are two distinct ovals O_p^1 and O_p^2, that are equivalent to O_p, and that intersect s times. Therefore as $s \to \infty$, $\#(O_p^1 \cap O_p^2) \to \infty$. ■

4 Oval Intersection Patterns

We want to look at all possible pairs of intersections of $O_x \cap O_y$ where O_x, O_y are ovals that pass though a vertex V on S. We can accomplish this by assuming V is a vertex of the master tile, fixing O_x and varying O_y, where x and y are edges intersecting at V. We denote the sequence of intersections between arbitrary ovals O_x, O_y around a vertex V with $\mathrm{stab}_G(V) = \langle v \rangle$ by the k-tuple:

$$V_{xy} = (\#(O_x \cap O_y), \#(O_x \cap vO_y), \ldots, \#(O_x \cap v^k O_y))$$

where

$$k = \left\{ \begin{array}{ll} \frac{o(v)-1}{2}: & o(v) \text{ is odd} \\ \frac{o(v)}{2} - 1: & o(v) \text{ is even} \end{array} \right\}.$$

If $x = y$ we replace $\#(O_x \cap O_y)$ by the number of vertices of O_x.

Remark 2 Pick v to rotate through the smallest possible angle ($\frac{2\pi}{l}$, $\frac{2\pi}{m}$, or $\frac{2\pi}{n}$ as appropriate). Then for any pair (O_x, O_y), $\langle v\rangle$-equivalent to another pair $(O_{x'}, O_{y'})$, the list $(\#(O_x \cap O_y), \#(O_x \cap vO_y), \ldots, \#(O_x \cap v^{o(v)-1}O_y))$ is simply a cyclic permutation of $(\#(O_{x'} \cap O_{y'}), \#(O_{x'} \cap vO_{y'}), \ldots, \#(O_{x'} \cap v^{o(v)-1}O_{y'}))$. Furthermore, $r_x(O_x) = O_x$ and $r_x(v^jO_y) = v^{-j}O_y$, so $\#(O_x \cap v^jO_y) = \#(O_x \cap v^{-j}O_y)$. It follows then that we need only consider the range on v^j given above and pick the x and y from representatives of the $\langle v\rangle$-equivalence classes of ovals passing through V. When v has even order there are two classes which may be taken to be the two edges of the tile meeting at V. When v has odd order there is only one class. In either case we can get all the information we need by considering V_{xx}, V_{xy} and V_{yy} where x and y are the two edges of the master tile meeting at V.

Definition 2 Consider the master tile T_0 and its bounding ovals O_p, O_q, and O_r. Then the <u>intersection sequence</u> is the set of the three vectors:

$$\vec{R} = (R_{qq}, R_{qp}, R_{pp}),\ \vec{P} = (P_{rr}, P_{rq}, P_{qq}), \text{ and } \vec{Q} = (Q_{pp}, Q_{pr}, Q_{rr}),$$

For notational convenience, we denote multiple intersection occurrences in exponential notation. For example $2, 2, 2, 2, 2$ is represented as 2^5.

Example 3 Consider the tiling of $(6, 3, 6)$-triangles by $G = Z_6$ with the generating triple (x, x^4, x) on a surface of $\sigma = 2$ with $\mu(x) = x^{-1}$. There are four vertices, P, Q, and R, and a^3P. The intersection sequence produced is:

$$\vec{R} = (4^3, 2^3, 2^3),\ \vec{P} = (4^2, 4^2, 4^2), \text{ and } \vec{Q} = (2^3, 2^3, 4^3)$$

We should note that depending on the parity of l, m, n we may record some repeats within our list. However, using this method in determining intersection patterns allows us to universally compare patterns on all tilings.

The ironic fact behind this analysis is that distinct tilings on distinct surfaces produced by distinct groups may have similar intersection sequences. For example, the triple (x^4, x, x^{11}) is a generating $(4, 16, 16)$ triple for Z_{16} on a surface with $\sigma = 6$ which produces this intersection sequence:

$$\vec{R} = (2^2, 1^2, 2^2),\ \vec{P} = (2^8, 1^8, 2, 1^3, 2, 1^3), \text{ and } \vec{Q} = (2, 1^3, 2, 1^3, 1^8, 2^8).$$

Also, (x^4, x, x^{19}) is a generating $(6, 24, 24)$ triple for Z_{24} on a surface of $\sigma = 10$ which produces this intersection sequence:

$$\overrightarrow{R} = (2^3, 1^3, 2^3);\ \overrightarrow{P} = (2^{12}, 1^{12}, 2, 1^3, 2, 1^3, 2, 1^3); \text{ and}$$
$$\overrightarrow{Q} = (2, 1^3, 2, 1^3, 2, 1^3, 1^{12}, 2^{12}).$$

Both Z_{16} and Z_{24} belong to a family of groups that produce "similar" intersection sequences on different surfaces. We can prove the following.

Proposition 4 If k is even and relatively prime to 5 and if $Z_{4k} = \langle x \rangle$ then $(a, b, c) = (x^4, x, x^{4k-5})$ is generating $(k, 4k, 4k)$ triple for a tiling on a surface of genus $\sigma = 2k - 2$ with $\mu(x) = x^{-1}$, and intersection sequence:

$$\overrightarrow{R} = (2^{\frac{k}{2}}, 1^{\frac{k}{2}}, 2^{\frac{k}{2}});\ \overrightarrow{P} = (2^{2k}, 1^{2k}, (2, 1^3)^{\frac{k}{2}});\ \overrightarrow{Q} = ((2, 1^3)^{\frac{k}{2}}, 1^{2k}, 2^{2k}).$$

Proof. Obviously $l = k$, $m = 4k$, $n = 4k$ are all even. Therefore $\lambda = \frac{k}{2}$, $\mu = 2k$, $\nu = 2k$. We also have $\langle a \rangle = \{x^4, x^8, \ldots, x^{4k} = 1\}$, $\langle b \rangle = Z_{4k}$, $\langle c \rangle = Z_{4k}$. We compute the length of all oval types.

$$\begin{aligned}
\|O_p\| &= 2 \cdot o((x^4)^{\frac{k}{2}}(x^{4k-5})^{2k}) = 2 \cdot o(x^{2k+8k-10k}) = 2. \\
\|O_q\| &= 2 \cdot o((x^4)^{\frac{k}{2}} x^{2k}) = 2 \cdot o(x^{4k}) = 2. \\
\|O_r\| &= 2 \cdot o(x^{2k}(x^{4k-5})^{2k}) = 2 \cdot o(x^{2k+8k-10k}) = 2.
\end{aligned}$$

R Vertex. There are $\frac{k}{2}$ O_p-type ovals and $\frac{k}{2}$ O_q-type ovals passing through the R vertex. Each oval has length 2. Every O_p-type oval will contain the same R vertex (since we are rotating about the R vertex on T_0), and one Q-type vertex. However, there is only 1 Q-type vertex in the entire tiling. Also, every O_q-type oval will contain the same R-type vertex, and one P-type vertex. Again, there is only 1 P-type vertex in the entire tiling. Hence, $\#(O_q \cap a^s O_q) = 2$, $\#(O_p \cap a^s O_p) = 2$, and $\#(O_q \cap a^s O_p) = 1$, yielding this partial intersection sequence:

$$\overrightarrow{R} = (2^{\frac{k}{2}}, 1^{\frac{k}{2}}, 2^{\frac{k}{2}})$$

P Vertex. There are $2k$ O_q and $2k$ O_r ovals passing through the P vertex. By Table 3, these ovals have these types of vertices:

- for $i = 0, \ldots, 2k-1$: $b^i O_r = \{\langle c \rangle, \langle b \rangle\}$

- for $j = 0, 4, 8, \ldots, 2k-4$: $b^j O_q = \{\langle a\rangle, \langle b\rangle\}$, $b^{j+1} O_q = \{x\langle a\rangle, \langle b\rangle\}$, $b^{j+2} O_q = \{x^2\langle a\rangle, \langle b\rangle\}$, $b^{j+3} O_q = \{x^3\langle a\rangle, \langle b\rangle\}$,

  Therefore we produce this partial intersection sequence around the P vertex.

$$\overrightarrow{P} = (2^{2k}, 1^{2k}, (2, 1^3)^{\frac{k}{2}})$$

Q Vertex. There are 2k O_p and 2k O_r ovals passing through the Q vertex. By Table 3, these ovals have these types of vertices:

- for $i = 0, \ldots, 2k-1$: $c^i O_r = \{\langle c\rangle, \langle b\rangle\}$
- for $j = 0, 4, 8, \ldots, 2k-4$: $c^j O_p = \{\langle a\rangle, \langle c\rangle\}$, $c^{j+1} O_p = \{x^{4k-5}\langle a\rangle, \langle c\rangle\}$, $c^{j+2} O_p = \{x^{4k-10}\langle a\rangle, \langle c\rangle\}$, $c^{j+3} O_p = \{x^{4k-15}\langle a\rangle, \langle c\rangle\}$,

  Therefore we produce this partial intersection sequence around the Q vertex.

$$\overrightarrow{Q} = ((2, 1^3)^{\frac{k}{2}}, 1^{2k}, 2^{2k})$$

∎

4.1 A conjecture and more examples

Producing examples of families of tilings with the similar oval intersection patterns is not difficult but tedious. You will notice that in the following conjecture and supporting examples the number of R, P, and Q-type vertices is the same. Also each tiling has the same oval type lengths and the same parity pattern.

Conjecture 1 Let $G_1 = \langle a_1, b_1, c_1\rangle$, and $G_2 = \langle a_2, b_2, c_2\rangle$ be tiling groups for two distinct tilings. Suppose that

$$\frac{|G_1|}{|\langle a_1\rangle|} = \frac{|G_2|}{|\langle a_2\rangle|}, \quad \frac{|G_1|}{|\langle b_1\rangle|} = \frac{|G_2|}{|\langle b_2\rangle|}, \text{ and } \frac{|G_1|}{|\langle c_1\rangle|} = \frac{|G_2|}{|\langle c_2\rangle|},$$

and that suppose G_1 and G_2 have the same parity pattern, and the same oval lengths. Then the oval intersection patterns in of the two tilings have the same pattern, i.e., the intersection pattern at corresponding vertices are cyclic repetitions of the same basic pattern.

Here are some additional examples (without proof) of families of groups that illustrate this predictable intersection sequence.

Example 4 Suppose that $\gcd(s, 3) = 1$ and set $G = Z_3 \times D_s = \langle w \rangle \times \langle r, f \rangle$. Then there is a $(2, 3s, 6)$ tiling by G, on a surface of genus $\sigma = s$ with involutary automorphism $\theta(x, y) = (x^{-1}, fyf)$, and generating triple $(a, b, c) = ((1, f), (w, r), (w^{-1}, r^{-1}f))$. The tiling produces the following intersection sequences.

If s is odd:

$$\overrightarrow{R} = (4, 2, 2s);$$
$$\overrightarrow{P} = (4, 2^{s-1}, 3, 2^{\frac{s-1}{2}}, 2^{\frac{s-1}{2}}, 3, 2^{s-1}, 4, 2^{\frac{s-1}{2}}, 4, 2^{s-1}, 3, 2^{\frac{s-1}{2}});$$
$$\overrightarrow{Q} = (2s, s^2, 1, 2, 1, 4, 2^{s-1}, 3, 2^{\frac{s-1}{2}} 4, 3^2).$$

If s is even:

$$\overrightarrow{R} = (4, 2, 2s);$$
$$\overrightarrow{P} = ((4, 2^{\frac{s}{2}-1})^3, 2^{\frac{3s}{2}}, 4, 2^{\frac{3s}{2}-1});$$
$$\overrightarrow{Q} = (2s, s^2, 2^3, 4^3).$$

Example 5 If k is odd, a $(2, k, 2k)$ tiling on a surface with $\sigma = \frac{k-1}{2}$ by Z_{2k} with $\theta(x) = x^{-1}$ and generators (x^k, x^{k-1}, x) produces an intersection sequence:

$$\overrightarrow{R} = (4, 2, 2);$$
$$\overrightarrow{P} = (4, 3^{\frac{k-1}{2}}, 3^{\frac{k-1}{2}}, 4, \; 4, 3^{\frac{k-1}{2}});$$
$$\overrightarrow{Q} = (2, 1^{k-1}, \; 1^{\frac{k-1}{2}}, 2, 1^{\frac{k-1}{2}}, 4, 3^{k-1}).$$

5 Further Questions

We conclude with these questions:

- Is the bound established by Theorem 1 every achieved? Theorem 1 gives an upper bound on the number of oval intersections for abelian tiling groups. However, an example of a tiling that reaches that bound has not been found. In fact, the highest observed number of intersections is 4, e.g., $G = Z_6$; $(a, b, c) = (x^4, x, x)$ and $G = Z_2 \times Z_6$; $(a, b, c) = ((x, 1), (x, y), (1, y^{-1}))$; both on a surface of genus $\sigma = 2$.

- Is there a construction that proves the result from Theorem 2 for non-abelian tiling groups that produce an OOO, or OOE tiling?

- Is there a characterization of groups that produce tilings where any two distinct intersecting ovals intersect the same number of times? Earlier we mentioned that any two distinct ovals on a sphere always intersect twice. There are other examples where every two ovals intersect in exactly the same number of points. Here is an example:

Example 6 Let $G = A_7$ (alternating group on 7 symbols) with generators $a = (1, 2, 3)$; $b = (1, 7, 6, 5, 4, 3, 2)$; $c = (3, 4, 5, 6, 7)$. If $f = (1, 3)(4, 7)(5, 6)$, then our automorphism is $\theta(x) = fxf$. There is a $(3, 7, 5)$ tiling by G on a surface with $\sigma = 409$ such that any two distinct intersecting ovals meet in exactly one point. The intersection pattern is given below. The 9's represent the intersection of an oval by itself.

$$\overrightarrow{R} = (9, 1, 1, 9, 9, 1);$$
$$\overrightarrow{P} = (9, 1, 1, 1, 1, 1, 1, 9, 9, 1, 1, 1);$$
$$\overrightarrow{Q} = (9, 1, 1, 1, 1, 9, 9, 1, 1).$$

6 Tables

Table 3 - Vertices of an Oval

Calculations are relative to the master tile: We use the generators $h_p; h_q;$ and h_r defined in Table 2. In each case the integer w varies as $0 \le w < o(h)$:

Case	Oval	Vertex Pattern	Vertices
OOO	O_p	RPQ	$h_p^w\langle a\rangle; h_p^w a^{\lambda+1}\langle b\rangle; h_p^w a^{\lambda+1}b^{\mu+1}\langle c\rangle$
	O_q	PQR	$h_q^w\langle b\rangle; h_q^w b^{\mu+1}\langle c\rangle; h_q^w b^{\mu+1}c^{\nu+1}\langle a\rangle$
	O_r	QRP	$h_r^w\langle c\rangle; h_r^w c^{\nu+1}\langle a\rangle; h_r^w c^{\nu+1}a^{\lambda+1}\langle b\rangle$
OOE	O_p	RPQPRQ	$h_p^w\langle a\rangle; h_p^w a^{\lambda+1}\langle b\rangle; h_p^w a^{\lambda+1}b^{\mu+1}\langle c\rangle; h_p^w a^{\lambda+1}b^{\mu+1}c^{\nu}\langle b\rangle;$
			$h_p^w a^{\lambda+1}b^{\mu+1}c^{\nu}b^{\mu}\langle a\rangle; h_p^w a^{\lambda+1}b^{\mu+1}c^{\nu}b^{\mu}a^{\lambda}\langle c\rangle$
	O_q	PQPRQR	$h_q^w\langle b\rangle; h_q^w b^{\mu+1}\langle c\rangle; h_q^w b^{\mu+1}c^{\nu}\langle b\rangle; h_q^w b^{\mu+1}c^{\nu}b^{\mu}\langle a\rangle$
			$h_q^w b^{\mu+1}c^{\nu}b^{\mu}a^{\lambda}\langle c\rangle; h_q^w b^{\mu+1}c^{\nu}b^{\mu}a^{\lambda}c^{\nu}\langle a\rangle$
	O_r	QPRQRP	$h_r^w\langle c\rangle; h_r^w c^{\nu}\langle b\rangle; h_r^w c^{\nu}b^{\mu}\langle a\rangle; h_r^w c^{\nu}b^{\mu}a^{\lambda}\langle c\rangle$
			$h_r^w c^{\nu}b^{\mu}a^{\lambda}c^{\nu}\langle a\rangle; h_r^w c^{\nu}b^{\mu}a^{\lambda}c^{\nu}a^{\lambda+1}\langle b\rangle$
EOE	O_p	RQ	$h_p^w\langle a\rangle; h_p^w a^{\lambda}\langle c\rangle$
	O_q	PQPR	$h_q^w\langle b\rangle; h_q^w b^{\mu+1}\langle c\rangle; h_q^w b^{\mu+1}c^{\nu}\langle b\rangle; h_q^w b^{\mu+1}c^{\nu}b^{\mu}\langle a\rangle$
	O_r	QPRP	$h_r^w\langle c\rangle; h_r^w c^{\nu}\langle b\rangle; h_r^w c^{\nu}b^{\mu}\langle a\rangle; h_r^w c^{\nu}b^{\mu}a^{\lambda}\langle b\rangle$
EEE	O_p	RQ	$h_p^w\langle a\rangle; h_p^w a^{\lambda}\langle c\rangle$
	O_q	PR	$h_q^w\langle b\rangle; h_q^w b^{\mu}\langle a\rangle$
	O_r	QP	$h_r^w\langle c\rangle; h_r^w c^{\nu}\langle b\rangle$

Table 4 - R-type vertices of O_p

Let $G = Z_t \times D_s = \langle w\rangle \times \langle r; f\rangle$ where $w^t = 1$ and $\langle a\rangle = \langle(1; f)\rangle = \{(1; f); (1; 1)\}$

O_p^1	$(1; 1)\langle a\rangle$	$(1; r)\langle a\rangle$	$(1; r^2)\langle a\rangle$	...	$(1; r^{s-1})\langle a\rangle$
O_p^2	$(w; f)\langle a\rangle$	$(w; fr)\langle a\rangle$	$(w; fr^2)\langle a\rangle$	...	$(w; fr^{s-1})\langle a\rangle$
O_p^3	$(w^2; 1)\langle a\rangle$	$(w^2; r)\langle a\rangle$	$(w^2; r^2)\langle a\rangle$	...	$(w^2; r^{s-1})\langle a\rangle$
O_p^4	$(w^3; f)\langle a\rangle$	$(w^3; fr)\langle a\rangle$	$(w^3; fr^2)\langle a\rangle$	...	$(w^3; fr^{s-1})\langle a\rangle$
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
O_p^t	$(w^{t-1}; 1)\langle a\rangle$	$(w^{t-1}; r)\langle a\rangle$	$(w^{t-1}; r^2)\langle a\rangle$	...	$(w^{t-1}; r^{s-1})\langle a\rangle$

References

[1] J. Cannon, Magma, computer algebra software. University of Sydney, john@maths.usyd.edu.au.

[2] A. Hurwitz, Algebraische Gebilde mit Eindeutigen Transformationen in sich, Math. Ann. 41 (1893), 403-441; Reprinted in Mathematische Werke I (Birkhauser, Basel, 1932), 392-436.

[3] S. Allen Broughton, Counting Ovals on a Symmetric Riemann Surface, Rose-Hulman Technical Report MSTR 97-04

[4] G. Gromadski, Groups of Automorphisms of Compact Riemann and Klein surfaces, Wyzsza Szkola Pedagogiczna w Bydgoszczy, (1993).

[5] D. Singerman, Automorphisms of Compact Non-orientable Riemann Surfaces, Glasgow Math. J., 12 (1971), 50-59.