Applications of Graph Theory to Separability

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Abstract

Let S be a surface with a triangular tiling Ψ. Let R be a reflection a side of one of the triangles; so that R is an orientation reversing isometry of the surface. Define $M = \{s \in S : Rs = s\}$. We then say that the surface S separates along the reflection R if $S - R$ has two components. This paper considers the applications of graph theoretic methods to determining whether a reflection is separating or not and compares the algorithmic efficiency of these methods to the current known methods.

Contents

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1 Introduction

1.1 Overview and Pictures

Let S be a compact orientable surface and let $\Psi = \{\triangle_0, \triangle_1, \ldots, \triangle_N\}$ be kaleidoscopic, geodesic tiling on the surface by the triangles $\triangle_0, \triangle_1, \ldots, \triangle_N$. We will define precisely what this means in a later section, but for now we will consider this to be the natural tiling of a surface where a high degree of symmetry is preserved as in Figure 1, Figure 2 and Figure 3. Furthermore, let R be an anti-conformal isometry of the surface of order 2. We will refer to R as a reflection of the tiling Ψ. Let $M = \{s \in S : Rs = s\}$ which shall be called the mirror of the reflection R. In this paper we consider whether the 2-manifold $S - M$ is a connected manifold, in other words we are considering the separability of the surface as defined by Bujalance and Singerman [5], Belk [1], Deblois, Baeth, and Powell [2], and Broughton [3]. We will discuss an alternative algorithm to Belk's Reflective Walk Algorithm for determining the separability of a surface and consider the potential theoretical contributions of this new algorithm.

Figure 1: The Icosahedral Tiling of the Sphere

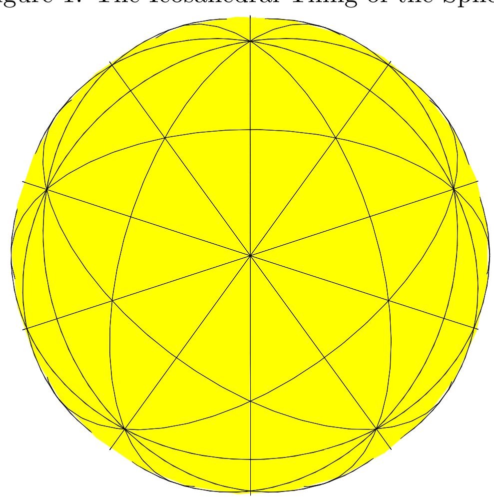


Figure 2: The $(2, 4, 4)$ Tiling of the Torus

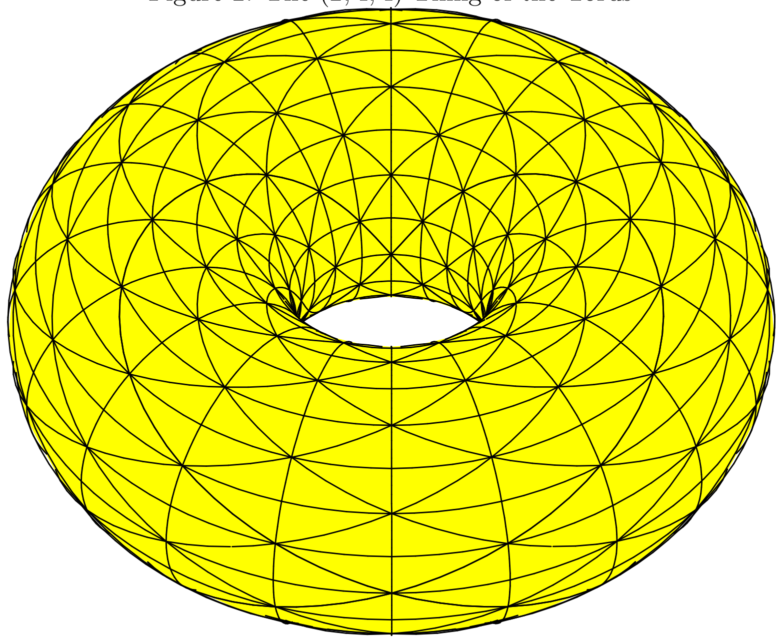
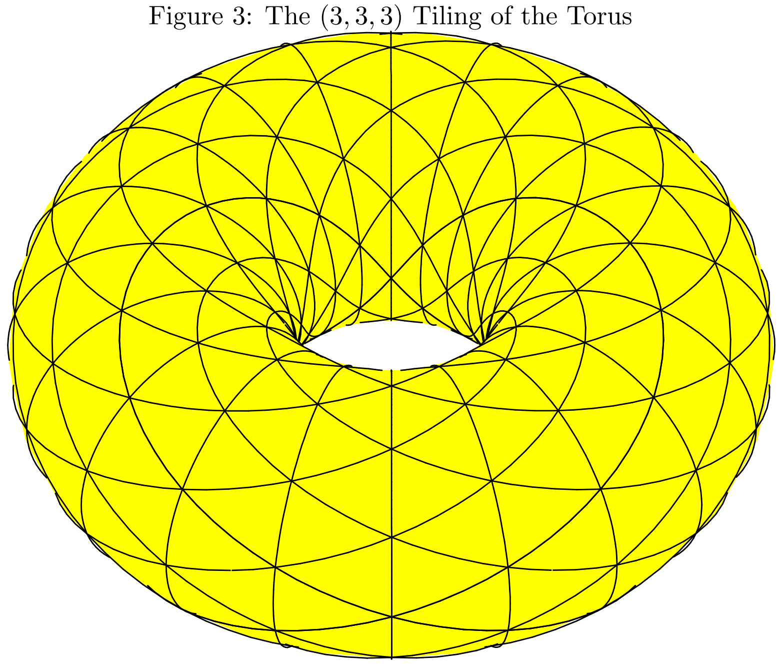


Figure 3: The $(3, 3, 3)$ Tiling of the Torus



1.2 Acknowledgments

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2 Tilings and their Groups

This section is derived from the notes for the "Tilings" REU program at Rose-Hulman Institute of Technology run by Dr. Broughton [4].

2.1 Tilings and Surfaces

For the purposes of this paper a surface S is a closed, compact, orientable 2-manifold residing in $\mathbb{R}^3$. In general a tiling of such a surface is a covering of the surface by non-overlapping polygons. Due to the rich algebraic structure on which we will elaborate later in this section, we will be primarily be dealing with tilings of the surface by congruent triangles. As such we can partially describe a tiling by the triple (l, m, n) where the angles of the triangle that tiles the surface are $(\frac{\pi}{l}, \frac{\pi}{m}, \frac{\pi}{n})$ respectively. Earlier we mentioned that we wanted to use tilings that have a high degree of symmetry, in order to do that we insist that the tiling be kaleidoscopic and geodesic.

Definition 1. *Kaleidoscopic* For each tile the natural reflection across any edge extends to a global reflection, R, of the surface S.

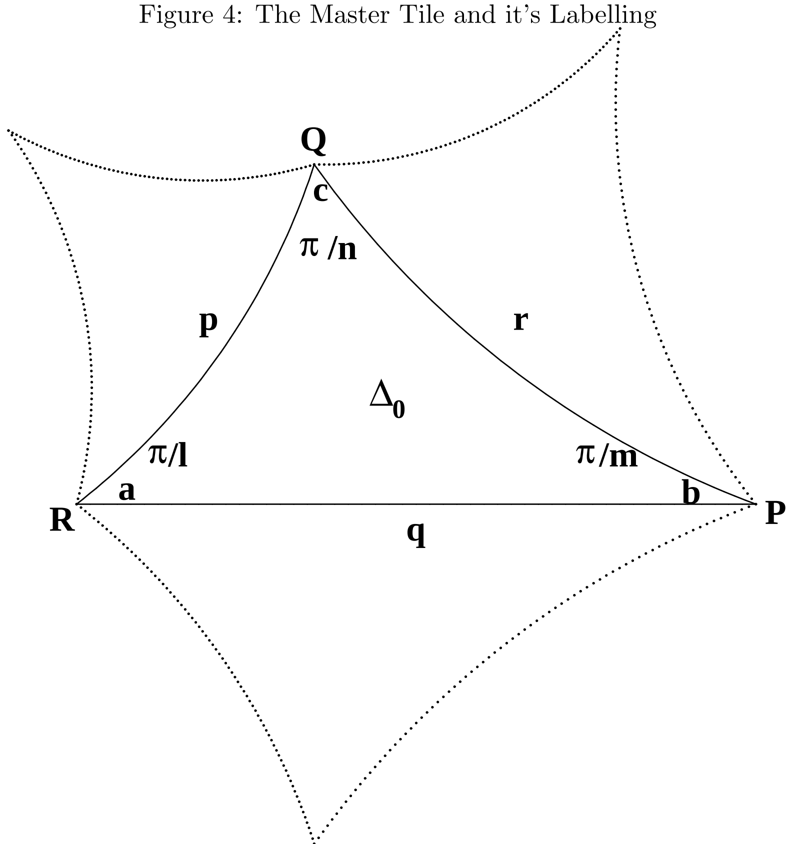
Definition 2. *Geodesic* Every edge of a given tile is part of a smooth closed curve on the surface consisting of edges of tiles.

2.2 Tiling Groups

There are two primary groups associated with a given tiling Ψ one is the conformal rotation group G (i.e., generated by rotations) and the other is the reflection group G^* (generated by anti-conformal reflections). To begin with we select some tile on the surface S, which we will call the master tile, and label this tile $\triangle_0$. As we shall see later, because of the transitive G^*-action on the tiles, any tile will do for the master tile.

In order to discuss the groups G and G^* on the surface S we first need to label our tile $\triangle_0$ as in Figure 4. Thus label the angles $\frac{\pi}{l}, \frac{\pi}{m}, \frac{\pi}{n}$, by a, b, c, respectively. Then label the edge between angles a and c with p, between a and b with q, and between b and c with r. Notice that the sides of the triangle are curved to represent the hyperbolic nature of the surface if the genus is 2 or greater.

Figure 4: The Master Tile and it's Labelling



Consider first the counter-clockwise rotation of a tile through angles of either, $\frac{2\pi}{l}$, $\frac{2\pi}{m}$, or $\frac{2\pi}{n}$ radians at a, b, and c, respectively. The natural labelling for these actions on the tiles are a, b, and c corresponding to the angle of rotation. If we repeatedly perform these operations on the master tile, it turns out that we can label exactly half of the tiles with a word consisting of a's, b's, and c's (since the angle of rotation is twice the angle of a triangle at that vertex. Clearly performing these rotations on any of the tiles so labelled takes one tile to another such labelled tile. Thus if we treat the master tile as the identity, it turns out that the action of a, b, and c on the tiling forms a group, namely $G = \langle a, b, c \rangle$, which is referred to as the *group of conformal rotations* on S arising from the tiling Ψ, and also as the *conformal tiling group*. In addition the generators of G satisfy the relations:

$$o(a) = l,\ o(b) = m,\ o(c) = n,\ abc = 1. \tag{1}$$

Now consider the reflections of the master tile across the edges p, q, and r. Again the natural labelling for the action of these reflections of the tiles are p, q, and r corresponding the edge over which the reflection occurs. Clearly by repeated reflection, every tile on S can be labelled with a word consisting of p's, q's, and r's. Furthermore, performing a reflection will obviously take such a labelled tile to another labelled tile. Again taking the master tile as the identity we get a group generated by (anti-conformal) reflections $G^* = \langle p, q, r \rangle$. The reflections satisfy these relations

$$p^2 = q^2 = r^2 = 1. \tag{2}$$

Also note that by reflecting across the q edge and then across the p edge we get the action of a on the master tile. Thus we have that

$$a = pq, \ b = qr, \ c = rp, \tag{3}$$

the latter two equations having similar explanations. Note that the last equation in (1) follows from (3) and (2) since $abc = pqqrrp = 1$.

Now we have that G is a normal subgroup of index 2 in G^* such that $\langle q \rangle \ltimes G = G^*$, as an internal semi-direct product. In fact, conjugation of the generators of G by q specifies an automorphism of G, θ, satisfying

$$\theta(a) = qaq^{-1} = qaq = qpqq = qp = a^{-1} \tag{4}$$

$$\theta(a) = qbq^{-1} = qbq = qqrq = rq = b^{-1}. \tag{5}$$

Thus the tiling is described by the group G and the triple (l, m, n). However, it turns out that there can be multiple tilings with the same group G and triple (l, m, n), so for a specific tiling both a and b need to be specified as well. We will need the following fact, which allows us to label the tiles with elements of G^*, as we have suggested we can do.

Proposition 1. *The tiling group G^* acts simply transitively on the tiles of S, and hence the elements of G^* are in 1-1 correspondence with the tiles in Ψ, via the map $g \to g\triangle_0$, whose inverse $\mathfrak{h} : g\triangle_0 \to g$ is a labelling map.*

Using this simple transitivity, we get some additional information about the surface S, including the number of tiles,

$$\# \text{ of tiles} = |G^*| = 2\,|G|\,, \tag{6}$$

and it's genus σ, through the Riemann-Hurwitz Equation:

$$\frac{2\sigma - 2}{|G|} = 1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} \right), \tag{7}$$

3 Reflective Walk Algorithm

In his 1999 paper [1], Jim Belk proposed a group theoretic algorithm that would determine the separability of a reflection in a given triangular tiling on the surface. The essence of this algorithm is repetitive reflection across the boundaries of some subset of the tiles without crossing the line of the reflection. First we will review the Reflective Walk Algorithm and it's proof and then introduce some of the underlying graph theory that allows this algorithm to work.

3.1 Group Theoretic Approach

In order to understand what makes the Reflective Walk Algorithm work on a group theoretic level we first need to prove some statements about a walk on the surface of a tiling.

Definition 3. *Reflective Walk* [Tiling] A sequence of tiles $\triangle_0, \ldots, \triangle_j$ determined by a sequence of reflections, $d_1, d_2, \ldots, d_j \in \{p, q, r\}$, via $\triangle_i = d_1 d_2 \cdots d_i \triangle_0$. The walk is usually represented as a string of p, q, and r, that moves the master tile across the surface of a tiling.

Remark 1. The constructed sequence of tiles $\triangle_i = d_1 d_2 \cdots d_i \triangle_0$, are such that $\triangle_i$ and $\triangle_{i-1}$ meet along an edge of type d_i. There is an associated path, to be discussed shortly, whose i'th edge is the hyperbolic line segment from the incenter of $\triangle_{i-1}$ to the incenter of $\triangle_i$ that crosses their common edge at right angles. The geometric walk is along this path. The walk in the group is the sequence of elements $g_0 = 1$, $g_1 = g_0 d_1 = d_1$, $g_2 = g_1 d_2 = d_1 d_2$, $\ldots$, $g_j = g_{j-1} d_j = d_1 d_2 \cdots d_j$, constituting a walk from $1 = g_0$ to $g = g_j$ in the group G^*. If we wish to construct a $d_1, d_2, \ldots, d_j$ walk starting at the tile $\triangle = g\triangle_0$ then the sequence of group elements is defined by $g_0 = g, g_i = g_{i-1} d_i, 1 \le i \le j$.

Remark 2. There is a 1-1 correspondence between sequences $g_0, \ldots, g_j$ such that $g_i^{-1} g_{i+1} \in \{p, q, r\}$, and sequences of tiles $\triangle'_0, \ldots, \triangle'_j$ such that $\triangle'_i$ and$\triangle'_{i+1}$ meet in an edge. The correspondence is $g_0, \ldots, g_j \to g_0\triangle_0, \ldots, g_j\triangle_0$. The correspondence is left G^*-equivariant, i.e., the sequence $gg_0, \ldots, gg_j$ maps to the sequence $gg_0\triangle_0, \ldots, gg_j\triangle_0$.

Remark 3. The following observation will be helpful in subsequent discussion. Suppose that $\triangle = g\triangle_0$. Then the reflections in the p, q, and r edges of $\triangle$ are gpg^{-1}, gqg^{-1}, and grg^{-1}, respectively.

These walks, and the simple observation that if a surface separates along a reflection R then there is no walk that can go from one component to the other without crossing the mirror of the reflection M, provide the heart of the group theoretic motivation of the Reflective Walk Algorithm.

Lemma 1 (See [1], [2]). *Let $\triangle_0$ be the master tile on the surface and let $g \in G^*$ such that $\triangle_j = g\triangle_0$. Then $\triangle_0$ and $\triangle_j$ are in the same component of S, after splitting along the mirror M of a reflection R, if and only if there are $d_1, d_2, \ldots, d_j \in \{p, q, r\}$ such that $g = d_1 d_2 \cdots d_j$ and*

$$d_1 d_2 \cdots d_i \neq R d_1 d_2 \cdots d_{i-1} \tag{8}$$

for all $1 \le i \le j$.

Proof. As in Remark 1, let $\triangle_i = d_1 d_2 \cdots d_i \triangle_0$. Then $(d_1 d_2 \cdots d_{i-1}) d_i (d_1 d_2 \cdots d_{i-1})^{-1}$ is the reflection in the d_i edge of $\triangle_{i-1} = d_1 d_2 \cdots d_{i-1} \triangle_0$. Thus

$$\begin{aligned}\triangle_i &= d_1 d_2 \cdots d_{i-1} d_i \triangle_0 \\ &= d_1 d_2 \cdots d_{i-1} d_i (d_1 d_2 \cdots d_{i-1})^{-1} d_1 d_2 \cdots d_{i-1} \triangle_0 \\ &= (d_1 d_2 \cdots d_{i-1}) d_i (d_1 d_2 \cdots d_{i-1})^{-1} d_1 d_2 \cdots d_{i-1} \triangle_0 \\ &= (d_1 d_2 \cdots d_{i-1}) d_i (d_1 d_2 \cdots d_{i-1})^{-1} \triangle_{i-1}.\end{aligned}$$

Hence the tiles $\triangle_{i-1}$ and $\triangle_i$ are separated by the mirror M, i.e., their common edge lies in M, if and only if the reflection $(d_1 d_2 \cdots d_{i-1}) d_i (d_1 d_2 \cdots d_{i-1})^{-1}$ equals R, since G^* acts simply transitively. So the tiles $\triangle_0$ and $\triangle_j$ are certainly in the same component if $d_1 d_2 \cdots d_i \neq R d_1 d_2 \cdots d_{i-1}$ for all $1 \leq i \leq j$. Now suppose that $g\triangle_0$ lies in the same component as $\triangle_0$. Then there is a path from the interior of $\triangle_0$ to the interior of $g\triangle_0$, not passing through any vertex, nor crossing M. Thus there is a series of tiles, $\triangle_0, \triangle_1, \ldots, \triangle_j = g\triangle_0$ such that $\triangle_i$ and $\triangle_{i-1}$ have a common edge not lying in M. By the previous discussion, $\triangle_i = (d_1 d_2 \cdots d_{i-1}) d_i (d_1 d_2 \cdots d_{i-1})^{-1} \triangle_{i-1}$ for some $d_i \in \{p, q, r\}$ and $g = d_1 d_2 \cdots d_j$. For this sequence of d_i's equation (8) holds. □

Lemma 2. *A tiling with a group G^* on a surface S does not separate along a reflection R of a tile $\triangle_0$ if and only if $|C| > |G|$ where C is the set of tiles reachable from $\triangle_0$ without crossing the mirror of the reflection R.*

Proof. Suppose that the tiling separates along the reflection R of the tile $\triangle_0$. Then surface has two disjoint connected components, that are reflections of each other. Thus these two components have the same number of tiles and since each tile must be in one of these components, each component must have $\frac{|G^*|}{2} = |G|$ tiles. Thus $|C| = |G|$. Suppose then that the tiling does not separate along the reflection R of the tile $\triangle_0$. Then the surface has a single component that has $|G^*|$ tiles. Thus $|C| > |G|$. □

Lemma 3. *A tiling separates along a reflection R of a tile $\triangle_0$ if and only if $R\triangle_0$ and $\triangle_0$ are not in the same component.*

Proof. Suppose that $R\triangle_0$ and $\triangle_0$ are not in the same component. Then they must be in different components and hence there must be at least 2 components and S separates along the reflection R. Conversely suppose that $R\triangle_0$ and $\triangle_0$ are in the same component. Then, if there are two components, $\triangle_0$ must be in one and $R\triangle_0$ must be in the other, since R interchanges the components. This is a contradiction. □

From Lemma 1, Lemma 2, and Lemma 3 we can develop the Reflective Walk Algorithm as developed by Belk [1] and refined by Deblois, Baeth, and Powell [2].

Reflective Walk Algorithm: *Group Theoretic*

1. Set $j = -1$, $V_j = \emptyset$, $V_{j+1} = \{1\}$.

2. Increment j by 1, set $W_j = V_j - V_{j-1}$, $W = \emptyset$.

3. Do the following:

   - For $g \in W_j$, if $gp \neq Rg$ then $W = W \cup \{gp\}$.
   - For $g \in W_j$, if $gq \neq Rg$ then $W = W \cup \{gq\}$.
   - For $g \in W_j$, if $gr \neq Rg$ then $W = W \cup \{gr\}$.

4. Set $V_{j+1} = V_j \cup W$.

5. If $|V_{j+1}| > |G|$, then S does not separate along R.

6. If $R \in V_{j+1}$, then S does not separate along R.

7. If $|V_{j+1}| = |V_j|$, then S separates along R.

8. Return to Step 2

The test performed in Step 6 is the modification of the Reflective Walk Algorithm suggested by Baeth, Deblois, and Powell [2]. Both the Reflective Walk Algorithm and the Modified Reflective Walk Algorithm can be easily implemented in computational algebra program MAGMA [16]. The implementation for these algorithms is available on the web at [17], in the MAGMA source files `IsSplit.mgm` and `IsSplitS.mgm` respectively. These implementations require input of G^* and the elements p, q, and r along with the desired reflection R. The author has produced additional code which uses these algorithms on the input of G, and (l, m, n) to determine the separability of all possible reflections and tilings given the specified group and triple. These are also available on the web at [17], in the MAGMA source files `Group.mgm` and `GroupS.mgm`.

3.2 Graph Theoretic Approach

We will now develop the Reflective Walk Algorithm from a primarily graph theoretic point of view. In order to do this we must first create a special graph on the surface of the tiling, alluded to in the previous section. This graph is called the dual of the tiling.

Definition 4. *Dual Graph of a Tiling* For each tile place a vertex at the incenter of the triangle and connect vertices whose tiles share an edge. [1]

Let δ be the dual graph of the tiling Ψ. As noted in the next remark, there is a bijective function between the set of vertices of δ, V_δ, and the tiles of Ψ. We denote the correspondence by $\mathfrak{f} : V_\delta \longrightarrow \Psi$, where $\mathfrak{f}(v)$ is the tile which contains v.

Remark 4. A graph γ embedded in a surface S defines a tiling or a 0-, 1-, and 2-dimensional cell decomposition of S, with vertices and edges defined in the obvious fashion. The faces or tiles of the γ-tiling will be the closures of the components of $S-\gamma$. We are only interested in the cases where the components of $S-\gamma$ are homeomorphic to open discs. For instance, the original tiling Ψ is the cell decomposition of S determined by the edges of the tiles of Ψ. The tiling determined by the dual graph δ is a tiling by regular polygons such that each original or Ψ-vertex is the center of a δ-face, which is a regular $2l$-gon, $2m$-gon or $2n$-gon; each Ψ-edge perpendicularly bisects a unique δ-edge; and each original tile or Ψ-face contains a unique δ-vertex. Thus the d-dimensional cells of Ψ are incident in a 1-1 fashion with the $2-d$ dimensional cells of δ, hence the name dual tiling.

For the cell decomposition of S induced by γ we let V_γ, E_γ and F_γ denote the sets of vertices, edges and faces respectively.

Definition 5. *Walk*[Graph Theoretic] A walk is a sequence of vertices $v_0v_1\cdots v_k$ such that there exists an edge between v_i and v_{i+1} for all $0\le i\le k-1$. [6]

Lemma 4. *The reflective walks on the surface of the tiling Ψ are in one to one correspondence with walks in δ.*

Proof. Suppose that $d_1d_2\cdots d_j$ is a walk on the surface of Ψ from $\triangle_0$ to $\triangle_j$ where $d_1,d_2,\ldots,d_j\in\{p,q,r\}$. We define $v_0=\mathfrak{f}^{-1}(\triangle_0)$. Inductively assume that $v_0,v_1,\ldots,v_i$ are defined, then $v_{i+1}=\mathfrak{f}^{-1}(\mathfrak{h}(\mathfrak{f}(v_i))d_{i+1}\triangle_0)$, where $\mathfrak{h}$ is the labelling map defined in Proposition 1. Note that this is the same as v_{i+1} being the incenter of the tile $\triangle_{i+1}=d_1d_2\cdots d_{i+1}\triangle_0$. Next we define W as a sequence of vertices in δ such that $W=v_0v_1v_2\cdots v_j$ where $v_0,v_1,v_2,\ldots,v_j\in V_\delta$. Since d_{i+1} takes the tile $\triangle_i$ to an adjacent tile $\triangle_{i+1}$ by $\triangle_{i+1}=\mathfrak{h}(\triangle_i)d_{i+1}\mathfrak{h}(\triangle_i)^{-1}\triangle_i$, i.e., crossing the common d_{i+1} edge, and since vertices of adjacent tiles are connected, then v_i and v_{i+1} are connected. Thus W is connected and hence a walk on δ. Define this function from walks on Ψ to walks on δ by $\mathfrak{W}$.

Suppose that $W=v_0v_1\cdots v_j$ is a walk in δ. Define $\triangle_0=\mathfrak{f}(v_0)$ and $\triangle_j=\mathfrak{f}(v_j)$. Note for all $0\le i\le j-1$, v_i and v_{i+1} are connected $\mathfrak{f}(v_i)$ and $\mathfrak{f}(v_{i+1})$ are adjacent, there must be an element $d\in\{p,q,r\}$ such that $(\mathfrak{h}(\mathfrak{f}(v_i))d)\mathfrak{f}(v_i)=\mathfrak{f}(v_{i+1})$, let this element be d_{i+1}. Note that this equation implies that $\mathfrak{f}^{-1}((\mathfrak{h}(\mathfrak{f}(v_i))d_{i+1})\mathfrak{f}(v_i))=v_{i+1}$, and hence this function is $\mathfrak{W}^{-1}$. Therefore $\mathfrak{W}$ is a bijection between walks on Ψ and walks on δ and hence the walks are in one to one correspondence. ☐

Note that because of this one-to-one correspondence between the reflective walks on Ψ and the walks on δ, we can define actions of G and G^* on the vertices of δ in a similar fashion as they act on the tiles of Ψ. Furthermore since $G\subset G^*$, we only need to specify this action in terms of G^*.

Definition 6. *Action of G^* on δ* Let v be a vertex of δ and let $g\in G^*$. Then $g\cdot v=\mathfrak{f}^{-1}(g\cdot\mathfrak{f}(v))$.

With this definition in place we can now talk about the nature of a reflection R on the surface of Ψ and how it relates to δ.

Lemma 5. *Let v_0 and v_j be vertices of δ. Then v_0 and v_j are in the same component if and only if there are vertices of δ, $v_1, v_2, \ldots, v_{j-1}$, such that $v_0 v_1 v_2 \cdots v_{j-1} v_j$ is a walk in δ and $v_i \neq \mathfrak{f}^{-1}(R\mathfrak{f}(v_{i-1}))$ for all $1 \leq i \leq j$.*

Proof. Let W be a walk from v_0 to v_j in δ. Then by Lemma 4 there is an equivalent walk on the surface from $\triangle_0 = \mathfrak{f}(v_0)$ to $\triangle_j = \mathfrak{f}(v_j)$, let this walk be $d_1 d_2 \cdots d_j$ where $d_1, d_2, \ldots, d_j \in \{p, q, r\}$. Then by Lemma 1, $\triangle_0$ and $\triangle_j$ are in the same component if and only if $d_1 d_2 \cdots d_i \neq R d_1 d_2 \cdots d_{i-1}$ for all $1 \leq i \leq j$. By Lemma 4 this is equivalent to $\mathfrak{f}(v_i) \neq R\mathfrak{f}(v_{i-1})$ for all $1 \leq i \leq j$. By applying $\mathfrak{f}^{-1}$ to both sides we get that this necessary and sufficient condition is $v_i \neq \mathfrak{f}^{-1}(R\mathfrak{f}(v_{i-1}))$ for all $1 \leq i \leq j$. □

With this theoretical background in place we can now interpret the Reflective Walk Algorithm in terms of it's underlying graph theory.

Reflective Walk Algorithm: *Graph Theoretic*

1. Create δ.
2. Remove all edges (v_i, v_j) such that $v_i = R\mathfrak{f}(v_j)$.
3. If resulting graph is connected, then S does not separate along R.
4. If resulting graph is not connected, then S separates along R.

4 Cayley Line Graph

In this section we develop the underling structure of an alternative algorithm to the Reflective Walk Algorithm. The motivation for this algorithm comes from an examination of a graph on the surface of the tiling. To begin with we provide the natural method of constructing this graph, and then look at the theoretical basis underlying this construction.

In the graph theoretic version of the Reflective Walk Algorithm, it seems apparent that a significant amount of times goes into the finding of the edges that cross the mirror in Step 2. In fact, if we examine the original Reflective Walk Algorithm, the iterative testing that goes on in Step 3 is precisely the testing required to ensure that the walk on the surface of Ψ does not cross M the mirror of the boundary. If there was a more efficient method of determining precisely the location of M in some representation of Ψ, there would be a more efficient means of determining the separability of the surface. In order to efficiently calculate the location of M, we will construct another graph, Γ, on the surface of Ψ.

Definition 7. *The line graph* Γ Place a vertex at the point of intersection of each edge of Ψ and the perpendicular edge from δ and then connect two vertices if their edges are part of the same tile of Ψ. Alternatively, the midpoints of edges could be chosen for the vertices.

In order to understand the motivation and theoretical basis for this definition, we first need to go back and understand the theoretical basis for the dual of the tiling, δ. Since δ is an attempt to bridge the gap between the group of the tiling and graph theory, we will first resort to Cayley Graphs to understand δ.

Definition 8. *Cayley Graph*[Directed] Let G be a group and $S \subseteq G$ such that $G = \langle S \rangle$. Then we define the Cayley Graph of G under S by assigning a vertex to each element of G and saying two vertices, v and w, are connected by an s colored edge directed edge from v to w if $vs = w$ where $s \in S$. [8]

Since we know that δ is an undirected, uncolored graph we turn to a modified definition of the Cayley Graph.

Definition 9. *Cayley Graph*[Undirected] Let G be a group and $S \subseteq G$ such that $G = \langle S \rangle$ and S is closed under inversion. Then we define the Cayley Graph of G under S by assigning a vertex to each element of G and saying two vertices, v and w, are connected if $vs = w$ for some $s \in S$. [15]

Theorem 1. *The graph δ is the undirected Cayley Graph of G^* generated by $\{p, q, r\}$.*

Proof. First note that $G^* = \langle p, q, r \rangle$ and that $p = p^{-1}$, $q = q^{-1}$, and $r = r^{-1}$: so the conditions of Definition 9 on S are satisfied. Let C be the undirected Cayley Graph of G^* generated by $\{p, q, r\}$. Note that since each group element corresponds to a unique tile and each tile corresponds to a vertex of δ there is a one-to-one correspondence between the vertices of C and the vertices of δ. Let this correspondence be $\mu = \mathfrak{h} \circ \mathfrak{f}^{-1} : \delta \longrightarrow C$. Let $v, w \in \delta$ such that there is an edge between v and w. Then since they represent adjacent tiles there is some element $s \in \{p, q, r\}$ such that $((\mathfrak{f}(v))s)\triangle_0 = \mathfrak{f}(w)$. Thus $\mu(v)$ is connected to $\mu(w)$.

Conversely suppose that x and y were vertices of C such that there was an edge between x and y. Then there would exist some element of $s \in \{p, q, r\}$ such that $xs = y$. Thus the tiles $x\triangle_0$ and $y\triangle_0$ meet along an edge in the tiling Ψ and hence $\mu^{-1}(x)$ is connected to $\mu^{-1}(y)$. Thus since edges in C are edges in δ and edges in δ are edges in C, μ is an isomorphism between C and δ, thus δ is the undirected Cayley Graph of G^* generated by $\{p, q, r\}$. $\square$

Now having found a theoretical basis for δ we compare it to our desired graph, Γ. If we lay both graphs over the surface of S we note that all the edges of δ go through vertices of Γ which leads us to another concept in graph theory, the line graph, which in turn will lead us to a theoretical understanding of Γ.

Definition 10. *Line Graph* The Line Graph, $L(G)$, of a graph $G = (V, E)$ is the graph whose vertices are in one to one correspondence with E. Furthermore two vertices in $L(G)$ are joined by an edge if and only if the corresponding edges in G share a vertex. [9]

Remark 5. As we shall detail presently, the Cayley line graph defines a G^*-equivariant cell decomposition or tiling of S which is related to the tiling Ψ as follows. By construction, the Γ-vertices are in $1-1$ correspondence with the Ψ-edges. There are $3|G^*|$ Γ-edges, three contained in each Ψ-tile. The Γ-faces are of two types. Each Ψ-tile properly contains a unique Γ-face. Each Ψ-vertex is surrounded by a Γ-face which is a $2l$-gon, $2m$-gon or $2n$-gon.

Theorem 2. *The Line Graph of the Cayley Graph of G^* generated by $\{p, q, r\}$, in other words $L(\delta)$, is Γ.*

Proof. First note that since each edge of δ crosses precisely one edge of the tiling, that the vertices of $L(\delta)$ can be identified with edges of tiles in Ψ. Thus the vertices of $L(\delta)$ and Γ are in one to one correspondence through some function $\lambda : L(\delta) \longrightarrow \Gamma$. Suppose that v and w were vertices in $L(\delta)$ that were connected. Then their respective edges e_v and e_w would have some common vertex x in δ. Since the edges in δ pass from one tile to another tile and both e_v and e_w are incident to x, the vertices v and w must border the same tile. Hence $\lambda(v)$ and $\lambda(w)$ are connected.

Suppose then that s and t were vertices in Γ such that they were connected. Then they must border the same tile, and hence there are edges e_s and e_t in δ that pass through the same edges as s and t and are incident to some common vertex. Thus $\lambda^{-1}(s)$ and $\lambda^{-1}(t)$ are connected in $L(\delta)$. Since edges in $L(\delta)$ are edges in Γ and edges in Γ are edges in $L(\delta)$, λ is an isomorphism between Γ and $L(\delta)$. Thus Γ is the Line Graph of the Cayley Graph of G^* generated by $\{p, q, r\}$. □

Definition 11. *Cayley Line Graph* By Theorem 2 we can refer to Γ as the Cayley Line Graph of G^*.

4.1 Theoretical Results

The introduction of the structure of the Cayley Line Graph provides a basis for further theoretical development of the properties associated with a tiling and relating to the separation of a tiling along a reflection. Here we present some results, both new and old, derived from the Cayley Line Graph.

Riemann-Hurwitz Equation Let Γ be the Cayley Line Graph of the tiling Ψ on the surface S. Since for each tile in Ψ has 3 edges crossing it the number of edges in Γ is $3\,|G^*|$. Furthermore since there are 3 vertices for each tile in G^* and each vertex is on the edge between two tiles, there are $\frac{3}{2}\,|G^*|$ vertices in Γ.

Definition 12. *ℓ-cycle* Given an vertex v and an edge e incident to it, the ℓ-cycle is the cycle formed by proceeding along e away from v until the next vertex w, and then taking the left most edge out of w relative to e. Repeat this process until reaching the vertex v. The path traced out forms the ℓ-cycle of v and e. Since Γ is locally planar and has no edge crossings, the left most edge is well defined on Γ.

Lemma 6. *For each of the edges e incident to a vertex v in the Cayley Line Graph Γ the ℓ-cycle exists.*

Proof. Let $v \in \Gamma$ be a vertex and let $e \in \Gamma$ be the edge (v, t). Furthermore let l be the leftmost edge out of t relative to e, $l = (t, s)$. If s, t, and v lie on the same tile then clearly the ℓ-cycle exists and is a 3-cycle. Suppose then that s, t, and v are not on the same tile. Then since v and t are connected and t and s are connected, s and v are on different tiles. However note that they share the same edge type. Let (s, r) be the left most edge out of s relative to edge l. Let $g \in G$ be the element that takes the vertex v to the vertex s. Thus $\mathit{tile}(s, r) = g \cdot \mathit{tile}(v, t)$ Furthermore $\overline{sr} = g \cdot \overline{vt}$ and hence two edge sections of the ℓ-cycle are formed by powers of g in the following form, $g^n \cdot \overline{vts}$, and the ℓ-cycle exists since g has finite order for all $g \in G$. □

Lemma 7. *A region f is a face of the embedding of Γ into S if and only if it's perimeter is an ℓ-cycle.*

Proof. Let f be a face of the embedding of Γ into S. Select some vertex v on the boundary of f and an edge e that is the left most edge of v relative to the face f. Then proceed to generate the ℓ-cycle from v and e, note that the leftmost edge will proceed along the boundary of f for each such v and e. Thus the boundary of any face of the embedding of Γ into S is an ℓ-cycle.

Let c be an ℓ-cycle in Γ. Suppose that r, the region enclosed by c is not a face. Then there exists some face a with ℓ-cycle p such that $a \subset r$. Since Γ is connected, p is connected to c by some path P. Since Γ is locally planar, $P \subset r \backslash a$. This contradicts c being an ℓ-cycle, so r is a face of Γ on S. □

Definition 13. The angle of an edge e in Γ is the angle between the edges associated with the endpoints of the e.

Lemma 8. *For each vertex v in Γ there are four ℓ-cycles, with two of them having cycle length 3, and the remaining two having cycle lengths of either $[2l, 2m]$, $[2l, 2n]$, or $[2m, 2n]$.*

Proof. First, since each vertex is on the edge between two tiles and in each of those tiles connects to two vertices bordering those tiles, each vertex is of degree 4, and hence there are precisely 4 ℓ-cycles for each vertex in Γ.

By the construction of Γ each vertex is going to be on two faces that have a perimeter of 3. By Lemma 7, these are ℓ-cycles and hence account for 2 of the 4 ℓ-cycles associated with that vertex.

Consider the two edges incident to a vertex v whose ℓ-cycles are not of length 3. Since neither of these edges can be to the immediate left of the other, and hence can not end the other's ℓ-cycle, the angle of these two edges is different. Note that each edge in these ℓ-cycles has the same angle, and hence the length of the cycles is $\frac{2\pi}{\angle \mathtt{edge}}$. Hence for any vertex $v \in \Gamma$, the length of the ℓ-cycles associated with it are either $[3, 3, 2l, 2m]$, $[3, 3, 2l, 2n]$, or $[3, 3, 2m, 2n]$. □

Lemma 9. *The number of faces in the embedding of* Γ *in* S *is*

$$|G^*|\,(1+\frac{1}{2l}+\frac{1}{2m}+\frac{1}{2n}).$$

Proof. Since all the faces in Γ are bordered by ℓ-cycles of length 3, $2l$, $2m$, or $2n$, $F = F_3 + F_{2l} + F_{2m} + F_{2n}$, where F is the total number of faces and F_n is the number of faces bordered by an ℓ-cycle of length n. Since for every tile there exits precisely one ℓ-cycle of length 3, $F_3 = |G^*|$. Consider the set of all vertices that form ℓ-cycles of length $2l$. All vertices except those between angle b and angle c will be on an ℓ-cycle of length $2l$, thus $|V_{2l}| = \frac{2}{3}|V|$, where V_{2l} is the set of vertices that lie on an ℓ-cycle of length $2l$. Thus since each $v \in V_{2l}$ is on precisely one ℓ-cycle of length $2l$ by Lemma 8 and there are $2l$ vertices on a ℓ-cycle of length $2l$,

$$\begin{aligned}
F_{2l} &= \frac{1}{2l}\frac{2}{3}|V| \\
&= \frac{1}{3l}\frac{3}{2}|G^*| \\
&= \frac{1}{2l}|G^*|\,.
\end{aligned}$$

Similarly $F_{2m} = \frac{1}{2m}|G^*|$ and $F_{2n} = \frac{1}{2n}|G^*|$. Hence

$$F = |G^*|\,(1+\frac{1}{2l}+\frac{1}{2m}+\frac{1}{2n}).$$

□

Theorem 3. *Riemann-Hurwitz Equation*

$$\frac{2\sigma-2}{|G|} = 1-(\frac{1}{l}+\frac{1}{m}++\frac{1}{n})$$

Proof. To begin with note that $\chi(S) = 2-2\sigma$ where σ is the genus of S. Furthermore since we know Γ can be embedded in S,

$$\begin{aligned}
\chi(S) &= V+F-E \\
2-2\sigma &= V+F-E \\
2-2\sigma &= \frac{3}{2}|G^*| + |G^*|\,(1+\frac{1}{2l}+\frac{1}{2m}+\frac{1}{2n}) - 3|G^*| \\
2-2\sigma &= 3|G| + |G|\,(2+\frac{1}{l}+\frac{1}{m}+\frac{1}{n}) - 6|G| \\
2-2\sigma &= |G|\,(\frac{1}{l}+\frac{1}{m}+\frac{1}{n}-1) \\
\frac{2-2\sigma}{|G|} &= \frac{1}{l}+\frac{1}{m}+\frac{1}{n}-1 \\
\frac{2\sigma-2}{|G|} &= 1-\frac{1}{l}-\frac{1}{m}-\frac{1}{n}
\end{aligned}$$

□

Harnack's Theorem [5] Let Γ be the Cayley Line Graph for the tiling Ψ of a surface S.

Definition 14. *Oval of a Mirror* An oval is one connected component of the mirror of a reflection

Lemma 10. *An oval θ goes through an equal number of Γ-vertices and Γ-faces.*

Proof. By definition we know that θ is a simple closed curve on the surface S. Since θ must contain an edge of the tiling Ψ, without loss of generality "start" θ on the Γ-vertex representing that edge. Since θ does not travel through the interior of a Ψ-tile, it must travel through a Γ-face before returning to a Γ-vertex. Thus for every vertex θ passes through exactly one face and vice versa. □

Definition 15. Let S be a surface with an embedded graph Γ such that S separates along some reflection R and let the associated mirror be M. Then define the surface S' by taking one component of the surface $S - M$ and for each oval in M closing the surface with an open unit disk. Likewise we define a graph Γ' on S' consisting of all edges and vertices on the closure of the selected component of $S - M$ and, in addition, the seams between the component and the unit disks, will become edges connecting adjacent Γ-vertices in the mirror.

Lemma 11. *The number of Γ'-vertices is $\frac{|V_\Gamma|+|V_M|}{2}$ where V_M is the set of Γ-vertices along the mirror of the reflection R.*

Proof. Since S' is formed by a reflection, it and it's reflection will have the same number of vertices. Since the Γ-vertices along the mirror will be counted twice, once for each component, the total number of vertices is $|V_M| + |V_\Gamma|$. Thus the number of Γ'-vertices is $\frac{|V_\Gamma|+|V_M|}{2}$. □

Lemma 12. *The number of Γ'-edges is $\frac{|E_\Gamma|}{2} + |V_M|$.*

Proof. Since S' and it's reflection will have the same number of edges, Γ' inherits $\frac{|E_\Gamma|}{2}$ edges from Γ. Furthermore, since there are additional edges from the seams, namely one edge for each face the mirror goes through, there are an additional $|V_M|$ edges in Γ' since the number of faces the mirror passes through is equal to the number of vertices the mirror passes through by Lemma 10. Thus the number of edges in Γ' is $\frac{|E_\Gamma|}{2} + |V_M|$. □

Lemma 13. *The number of Γ'-faces in S' is $\frac{|F_\Gamma|+|V_M|}{2} + |\Theta|$.*

Proof. As in the above arguments, there are a equal number of Γ'-faces in S' and it's reflection. Furthermore, since each of the faces along the ovals were split into two faces by the new edges, there are an additional $|V_M|$ faces available. Finally, adding to that the faces created by the unit disks attached to each oval and we have that the total number of Γ'-faces in S' is $\frac{|F_\Gamma|+|V_M|}{2} + |\Theta|$. □

Definition 16. The genus of S' is $\sigma_{S'}$ and the genus of S is σ_S.

Lemma 14.

$$\sigma_{S'} \leq \left\lfloor \frac{\sigma_S}{2} \right\rfloor$$

Proof. This follows immediately from the next Lemma. □

Lemma 15.

$$\sigma_S - 2\sigma_{S'} = |\Theta| - 1$$

Proof.

$$\begin{aligned}
\chi(S') &= |V_{\Gamma'}| + |F_{\Gamma'}| - |E_{\Gamma'}| \\
2 - 2\sigma_{S'} &= |V_{\Gamma'}| + |F_{\Gamma'}| - |E_{\Gamma'}| \\
2 - 2\sigma_{S'} &= \frac{|V_\Gamma| + |V_M|}{2} + \frac{|F_\Gamma| + |V_M|}{2} + |\Theta| - \frac{|E_\Gamma|}{2} - |V_M| \\
2 - 2\sigma_{S'} &= \frac{|V_\Gamma| + |F_\Gamma| - |E_\Gamma|}{2} + |\Theta| \\
2 - 2\sigma_{S'} &= 1 - \sigma_S + |\Theta| \\
\sigma_S - 2\sigma_{S'} &= |\Theta| - 1
\end{aligned}$$

□

Corollary 1 (Lemma 15). *If the genus of S is odd then $|\Theta| \geq 2$.*

Proof. By Lemma 15 $\frac{1+\sigma_S-|\Theta|}{2} = \sigma_{S'}$ and by Lemma 14 $\sigma_{S'} \leq \left\lfloor \frac{\sigma_S}{2} \right\rfloor$. Thus $1-|\Theta| \leq 2\left\lfloor \frac{\sigma_S}{2} \right\rfloor - \sigma_S$ and $|\Theta| \geq \sigma_S - 2\left\lfloor \frac{\sigma_S}{2} \right\rfloor + 1$. If σ_S is odd, then $\sigma_S - 2\left\lfloor \frac{\sigma_S}{2} \right\rfloor = 1$. Hence $|\Theta| \geq 2$. □

Corollary 2 (Lemma 15).

$$\sigma_S - |\Theta| \equiv 1 \pmod 2$$

Proof. By Lemma 15 $\sigma_S - |\Theta| = 2\sigma_{S'} - 1$, thus since $\sigma_{S'}$ is integer, $\sigma_S - |\Theta| \equiv 1 \pmod 2$. □

Theorem 4. *Harnack's Theorem*
If the surface S of genus σ_S separates along the reflection R that induces the set of ovals Θ then $1 \leq |\Theta| \leq \sigma_S + 1$.

Proof. Clearly by the definition of Θ and R, $|\Theta| \geq 1$. Since R separates S, by Lemma 15, $\sigma_S - 2\sigma_{S'} = |\Theta| - 1$ and hence $\sigma_S + 1 - |\Theta| = 2\sigma_{S'} \geq 0$. Thus $\sigma_S + 1 \geq |\Theta|$. □

Minimal Length Separating Mirror By using ideas of the connectivity of a graph as applied to Γ we can determine the minimal number of edges a mirror must contain in order to separate.

Lemma 16. *The degree of every vertex in Γ is 4.*

Proof. First note that every vertex lies on an edge between two tiles in Ψ. Let v be a vertex in Γ. By definition v is only connected to vertices that lie on the same Ψ-tile edges as v. Since v lies on two tiles and there are 2 non-v edges to these tiles, v is connected to 4 other vertices hence has degree 4. Because v was arbitrary every vertex in Γ has degree 4. □

Definition 17. The vertex connectivity of a graph is the minimal number of edges needed to be removed in order to disconnect the graph. [9]

Theorem 5. *A mirror M of a reflection must contain at least 4 edges in order to separate the surface S.*

Proof. By a well known result in graph theory and Lemma 16, the vertex connectivity of Γ is at most 4. [10] Suppose that the vertex connectivity was 3, then there would exist 3 vertices that could be removed to isolate another vertex. This would correspond to 3 edges in Ψ being removed to isolate a fourth edge. Since these edges isolate an edge they must be connected, and hence form a triangle isolating a fourth edge, a contradiction. Thus the vertex connectivity of Γ is 4. Furthermore since cutting along an edge in Ψ corresponds to removing a vertex in Γ the minimum number of edges a mirror must contain to separate the surface is 4. □

4.2 Cayley Line Graph Algorithm

The Cayley Line Graph Algorithm is an attempt to work primarily in an induced graph of the tiling Ψ rather than in the induced group of the tiling G^* as the Reflective Walk Algorithm does. In order to formulate such an algorithm certain theoretical connections need to be made between the action of a reflection on the surface and the induced action of the reflection on the graph Γ.

Lemma 17. *A reflection R on the surface of S induces an automorphism of the graph Γ whose fixed points lie along the mirror of the reflection M.*

Proof. Begin by embedding Γ on the surface as in it's definition. Since R is an isometry of the surface, preserving the tiling, by performing R on the surface we take Γ-edges to Γ-edges and Γ-vertices to Γ-vertices. As incidence is preserved, this clearly is an automorphism of Γ. Note that since R fixes only those points along the mirror of the reflection, the fixed points of the induced automorphism will lie along the mirror of the reflection. □

Lemma 18. *A surface S separates along a reflection R if and only if $\Gamma - \texttt{Fix}(R)$ is not connected, where $\texttt{Fix}(R)$ are the fixed points of the automorphism of Γ induced by R.*

Proof. Suppose that S separates along R and $\Gamma - \texttt{Fix}(R)$ is connected. Then there exists some path P in $\Gamma - \texttt{Fix}(R)$ between any two vertices v and w. Suppose that v and w are on separate components of S after the separating along R. Then there would be some path that crossed the mirror M of R. Since all the vertices along the mirror were fixed by R, this path must cross the mirror via an edge. This is a contradiction of the construction of Γ since no edge of Γ crosses an edge of the tiling.

Conversely, suppose that S does not separate along R. We know then by the Graph Theoretic Reflective Walk Algorithm, that δ is still connected, after removing the edges crossing M, since this action is equivalent to removing the fixed points of the reflection in Γ. By a result of graph theory [9], the line graph of a graph is connected if and only if the graph was connected. Thus $\Gamma - \mathtt{Fix}(R)$ is connected. Thus a surface is connected after a reflection R if and only if $\Gamma - \mathtt{Fix}(R)$ is connected. □

With this background in place we can now explicitly state the Cayley Line Graph Algorithm.

Cayley Line Graph Algorithm

1. Construct Γ.
2. Determine the fixed points under R.
3. Remove fixed points.
4. If resulting graph is connected, S does not separate under R.
5. If resulting graph is not connected, S separates under R.

Algorithmic Details Unfortunately this general algorithm is not detailed enough to be efficiently implemented. Specifically the methods of implementing Step 1 and Step 2 will greatly effect the overall efficiency of the algorithm. To begin with we will present a method for constructing Γ.

Lemma 19. *Every edge in Ψ borders exactly one tile of the form $g \cdot \triangle_0$ where $g \in G$ and exactly one other tile, which has the form $gs \cdot \triangle_0$, $s \in \{p, q, r\}$.*

Proof. Let the edge be e and let r_e be the reflection in e. Note that $r_e \in G^* - G$. By simple transitivity, there is a g' such that e lies on $g' \cdot \triangle_0$. If $g' \in G$ set $g = g'$, $h = r_e g'$, else $g = r_e g'$, $h = g'$. The uniqueness follows from simple transitivity. Finally, note that $r_e = gsg^{-1}$ for some $s \in \{p, q, r\}$ and so $h = r_e g = gsg^{-1} g = gs$. □

Lemma 20. *Let $g, h, i \in G$ and $k \in G^* - G$ be the elements such that $g \cdot \triangle_0$, $h \cdot \triangle_0$, $i \cdot \triangle_0$ surround $k \cdot \triangle_0$, and that $g \cdot \triangle_0$ and $k \cdot \triangle_0$ share the p-edge, $h \cdot \triangle_0$ and $k \cdot \triangle_0$ share the q-edge, and $i \cdot \triangle_0$ and $k \cdot \triangle_0$ share the r-edge. Then*

$$g = kp, h = kq, \text{ and } i = kr,$$
$$h = ga, i = hb, \text{ and } g = ic.$$

Proof. Because of their locations relative to k, $gp = k$, $hq = k$ and $ir = k$. Thus

$$\begin{array}{ccc} gp = k = hq & hq = k = ir & ir = k = gp \\ gp = hq & hq = ir & ir = gp \\ gpq = h & hqr = i & irp = g \\ ga = h & hb = i & ic = g \end{array}$$

□

The implications of Lemma 19 and Lemma 20 are that we can create an algorithm for constructing Γ without resorting to developing all of G^* and without having to go through the Cayley Construction. Hence we solve the algorithmic problem posed by Step 1 with this algorithm:

The G-Construction of Γ

1. Create 3 sets of vertices of size $|G|$, labelled by the elements of G, say $V_p = G \times \{p\}$, $V_q = G \times \{q\}$, and $V_r = G \times \{r\}$.
2. Set $V_\Gamma = V_p \cup V_q \cup V_r = G \times \{p, q, r\}$.
3. For $g \in G$ connect any two Γ-vertices that lie on the boundary of $g \cdot \triangle_0$, i.e., add $((g,p),(g,q))$, $((g,p),(g,r))$, and $((g,q),(g,r))$ to the edge set E_Γ.
4. For $k = gq \in G^* - G$ connect any two Γ-vertices that lie on the boundary of $k \cdot \triangle_0$, i.e., add $((kp,p),(kq,q)) = ((ga^{-1},p),(g,q))$, $((kp,p),(kr,r)) = ((ga^{-1},p),(gb,r))$, and $((kq,q),(kr,r)) = ((g,q),(gb,r))$ to the edge set E_Γ.

The following characterization of E_Γ is sometimes useful.

$$E_\Gamma = \{((g,s),(h,t)) : g, h \in G, s, t \in \{p,q,r\}, g = h \text{ or } gs = ht\} \tag{9}$$

The next algorithmic detail that needs to be cleared up is the tricky matter of determining the fixed points of the reflection. The primary difficulty in this matter is the development of the automorphism of Γ based solely on the reflection and the action of G on the tiles. In order to efficiently calculate the fixed points we need to make some theoretical observations about the nature of the automorphism induced by the reflection.

Lemma 21. *The elements of G^* induce graph automorphisms of Γ, which is easily calculated in terms of the group action. A reflection $R \in \{p,q,r\}$ induces G-automorphism on V_R.*

Proof. It is clear that every element of $h \in G^*$ determines an automorphism of Γ, by the G^*-action on Ψ, we just need to determine the format. If $h \in G$ then $g \cdot \triangle_0$ is mapped to $hg \cdot \triangle_0$ and so the mapping on Γ-vertices is $(g,s) \to (hg,s)$ for $s = p, q, r$. It is clear from the characterization (9) that this map is a graph automorphism. Next, suppose that $h = R \in G^* - G$. Then, as $g \cdot \triangle_0$ is mapped to $Rg \cdot \triangle_0$ the vertex (g, s) should map to (Rg, s), but Rg does not lie in G. By Lemma 20 the properly labelled vertex is (Rgs, s). Again from the characterization (9) we see that the map is a graph automorphism.

When $R = p, q, r$ we can write the action in terms the group G and the automorphism θ. By the construction of G^* from G, the conjugation automorphism induced by q is the automorphism θ on G satisfying, $\theta(a) = a^{-1}$ and $\theta(b) = b^{-1}$. If $R = q$ then the map on vertices is $(g, s) \to (\theta(q)qs, s)$. In particular, q acts on V_q in the same way that θ acts on G, q acts on V_p by $g \to \theta(g)a^{-1}$ and q acts on V_r by $g \to \theta(g)b$. Furthermore, the automorphisms of G, γ and τ, induced by $p = aq$ and $r = qb$ satisfy $\gamma(a) = a^{-1}$, $\gamma(c) = c^{-1}$, and $\tau(b) = b^{-1}$,$\tau(c) = c^{-1}$. These automorphisms of G work on the elements of V_R in a similar fashion. Specifically we have the following table for the actions of p, q, r on V_p, V_q, V_r.

	$(g,p) \in V_p$	$(g,q) \in V_q$	$(g,r) \in V_p$
p-action on g	$\gamma(g)$	$\gamma(g)a$	$\gamma(g)c^{-1}$
q-action on g	$\theta(g)a^{-1}$	$\theta(g)$	$\theta(g)b$
r-action on g	$\tau(g)c$	$\tau(g)b^{-1}$	$\tau(g)$

□

Lemma 22. *The following relationships hold for the automorphisms induced by a reflection.*

$$\begin{aligned}
\theta(x) &= \gamma(axa^{-1}) \\
\tau(x) &= \gamma(c^{-1}xc) \\
\gamma(x) &= \theta(a^{-1}xa) \\
\tau(x) &= \theta(bxb^{-1}) \\
\gamma(x) &= \tau(cxc^{-1}) \\
\theta(x) &= \tau(b^{-1}xb)
\end{aligned}$$

Proof. Since $pq = a$ then $q = pa$ and hence $\theta(x) = qxq^{-1} = paxa^{-1}p = \gamma(axa^{-1})$. The other formulas are similar. □

Determination of Automorphism

1. Using the appropriate generating mapping from Lemma 21 (and the table in the proof) create the appropriate automorphism of G.

2. Given the γ, θ, or τ from Step 1, use the relations from Lemma 22 to determine the total automorphism on Γ.

With these efficient implementations of Step 1 and Step 2 of the Cayley Line Graph Algorithm, the algorithm can be coded into MAGMA [16] for testing. A MAGMA source file implementing this version of the Cayley Line Graph

Algorithm is available online at [17] in the file `Graph.mgm`. This program takes the group G and the orders of a, b, and c and finds the separability for all non-automorphic sets of generators and a few automorphic ones. This compromise was based on attempting to optimize for speed, the effort to remove the few automorphic cases that are processed was considered to be less than the effort to calculate the separability. A similar decision was made in `Group.mgm` and `GroupS.mgm`.

Remark 6. By the nature of the theoretic creation of Γ the Cayley Line Graph Algorithm in it's current form can easily be applied to tilings by polygons other than triangles, with few adjustments to the process of creating Γ.

5 Comparison

5.1 Discussion

Determining a precise worst case of analysis of the three algorithms (`Group.mgm`, `GroupS.mgm`, and `Graph.mgm`), would involve an analysis that takes into account the group efficiency of group operations for varying groups, a thing that is tricky at best. We decided to resort to a purely computational comparison. In order to gather data for a computational comparison, the algorithms were timed on a set of over 2000 tilings [7], with repetition, on a dedicated machine. The computations for this analysis were run on a Sun Ultra 10, 360 Mhz processor with 256M of RAM, using MAGMA V2.7-3 [16].

The first step in our analysis of the empirical running times is to determine whether the Reflective Walk Algorithm or the Modified Reflective Walk Algorithm is faster. To do this we first plot all the data points for both algorithms in Figure 5. (All figures for this discussion are in the next subsection.) As the diagram makes clear there is little discernable difference, except isolated locations such as the groups of order 504, where the original algorithm is clearly faster. In Figure 6 we zoom in a bit to reveal some finer differentiation, and we see that again there is little discernable difference.

In hopes of making a finer distinction between the original and the modified algorithm, we plot the differences between the two times for all the data points in Figure 7. In this graph the reason for the difficulty of discerning the difference between the two becomes clear as the data points cluster in both the positive and negative time ranges. However there seems to be slightly more negative numbers than positive, leading us to believe that the original algorithm is slightly faster on the whole.

In order to verify this conclusion and to clear up some of the scattering we compare the differences in the average time for a given $|G|$ in Figure 8. Again we see that the difference are hardly noticeable but we can see that there is a definite general trend towards the original Reflective Walk Algorithm being faster than the Modified Reflective Walk Algorithm.

Having decided that the original Reflective Walk Algorithm is barely faster than the Modified Reflective Walk Algorithm, we turn now to comparing the original Reflective Walk Algorithm with Cayley Line Graph Algorithm. The first step in our analysis is to get a general feel for the lay of the data. In order to do this we plot both the times of the original Reflective Walk Algorithm and the Cayley Line Graph Algorithm in Figure 9. Although the plot has a significant amount of scatter it seems apparent that the Cayley Line Graph Algorithm is significantly faster than the original Reflective Walk Algorithm.

In order to get a better feel for the relationship between the timing of the two algorithm we plot the ratio of the time for the Cayley Line Graph Algorithm over the time for the original Reflective Walk Algorithm in Figure 10. Again we see that the majority of the data points fall below 1, implying that the Cayley Line Graph Algorithm is indeed faster. In fact the diagram seems to indicate that as $|G| \longrightarrow \infty$, the ratio decreases exponentially.

In order to clarify this conclusion we plot the ratio of the average times by group order in Figure 11. This diagram confirms our suspicions that the Cayley Line Graph Algorithm is significantly faster than the original Reflective Walk Algorithm, exponentially so as the group size gets large. Furthermore from close examination of the data over the small group sizes, where the original Reflective Walk Algorithm is occasionally faster that the Cayley Line Graph Algorithm, it is apparent that whatever speed gains achieved by the original Reflective Walk Algorithm due to smallness are minute, to the point of being practically unmeasurable time gains. Thus we would conclude that in general, the Cayley Line Graph Algorithm is the most efficient method of determining separability currently available, however as the rapid growth of the times as the groups get large and complex leaves significant room for improvement over this algorithm.

Remark 7. The wide degree of scattering in the timing data, reveals the inadequacy of group size as a measure of group complexity. In an ideal world the group size and inherent complexity of the group would be combined in some sort of measure that would give a smoother growth pattern in both the Reflective Walk Algorithm and the Cayley Line Graph Algorithm. This follows with the observations of Broughton [14] on the difficulties of generalizing the methods of [3] to a cohesive, universal theory of the separability of the surface. In particular the influences of group complexity on the difficulties of determining separability of a surface.

5.2 Comparison Figures

Figure 5: Comparison of `Group.mgm` and `GroupS.mgm`

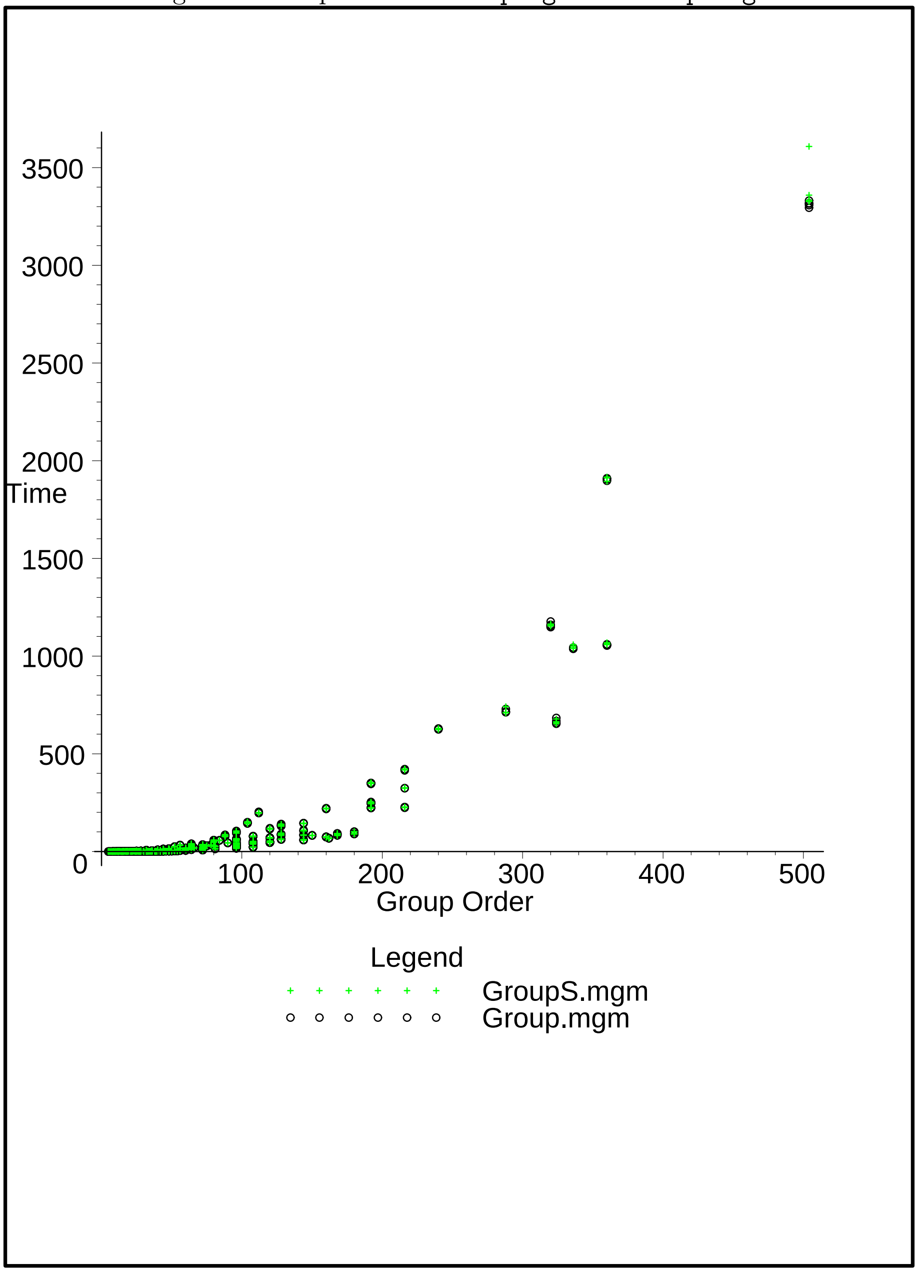


Figure 6: Comparison of `Group.mgm` and `GroupS.mgm` (Zoomed)

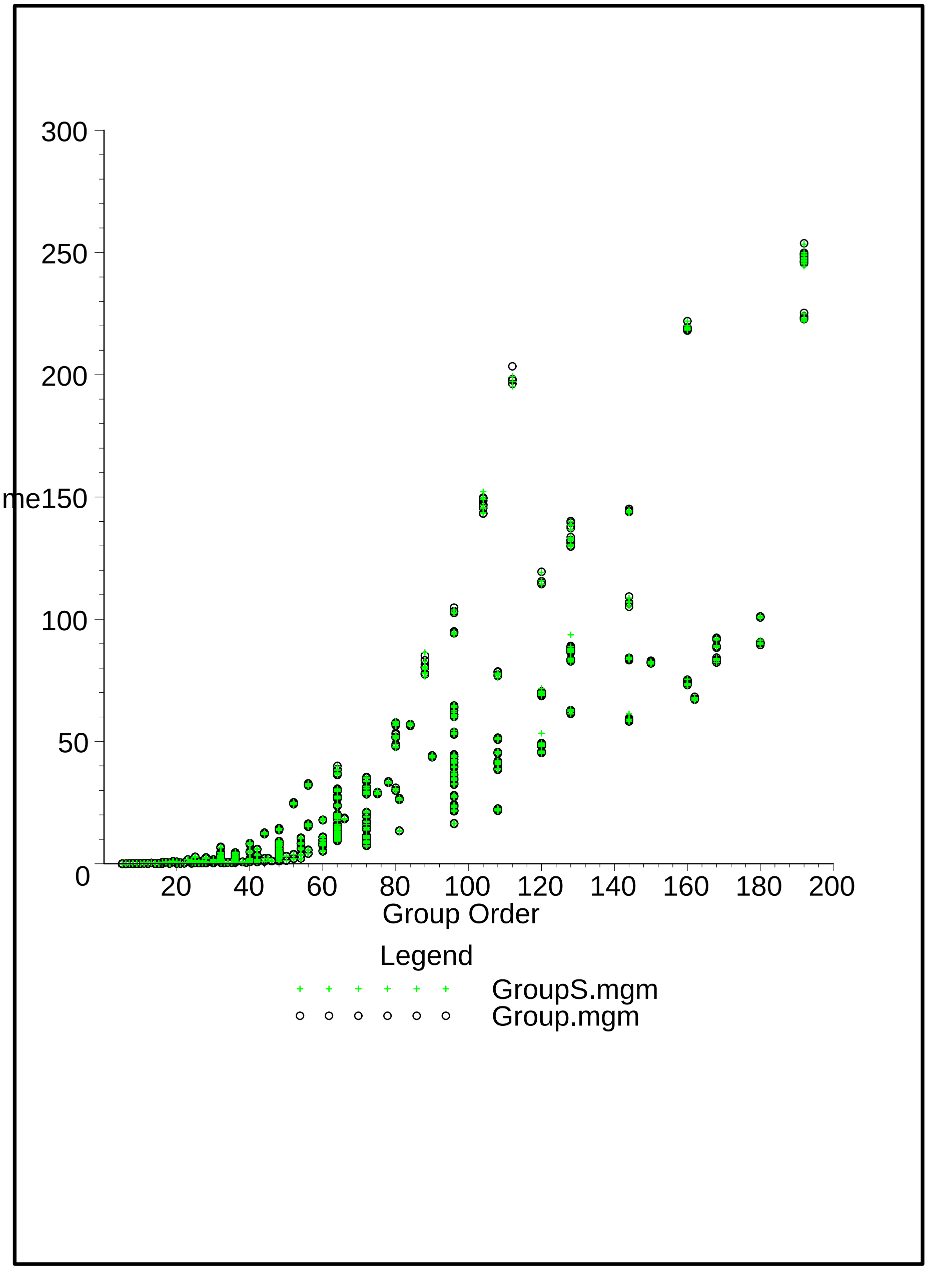


Figure 7: `GroupS.mgm` − `Group.mgm`

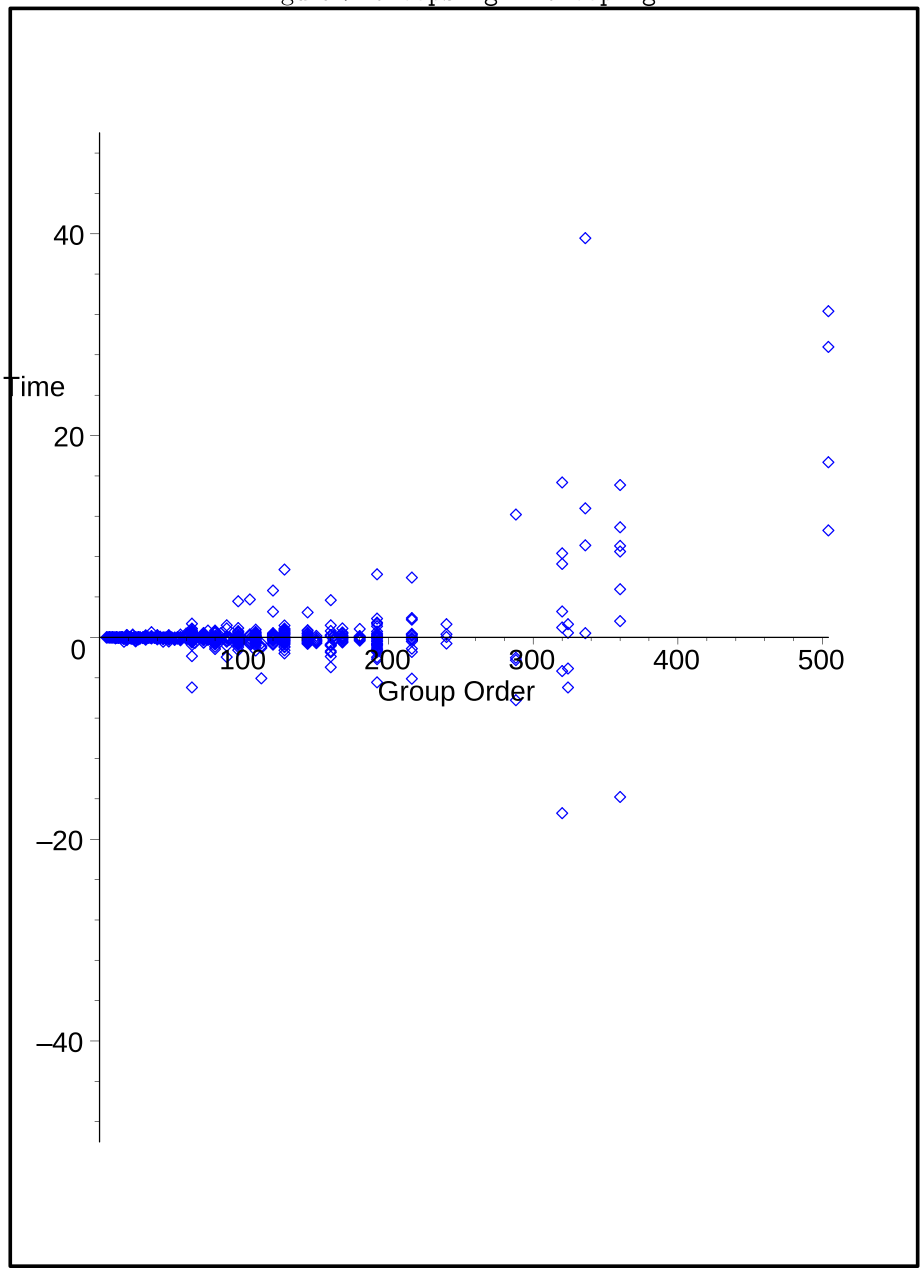


Figure 8: $\overline{\texttt{GroupS.mgm}} - \overline{\texttt{Group.mgm}}$

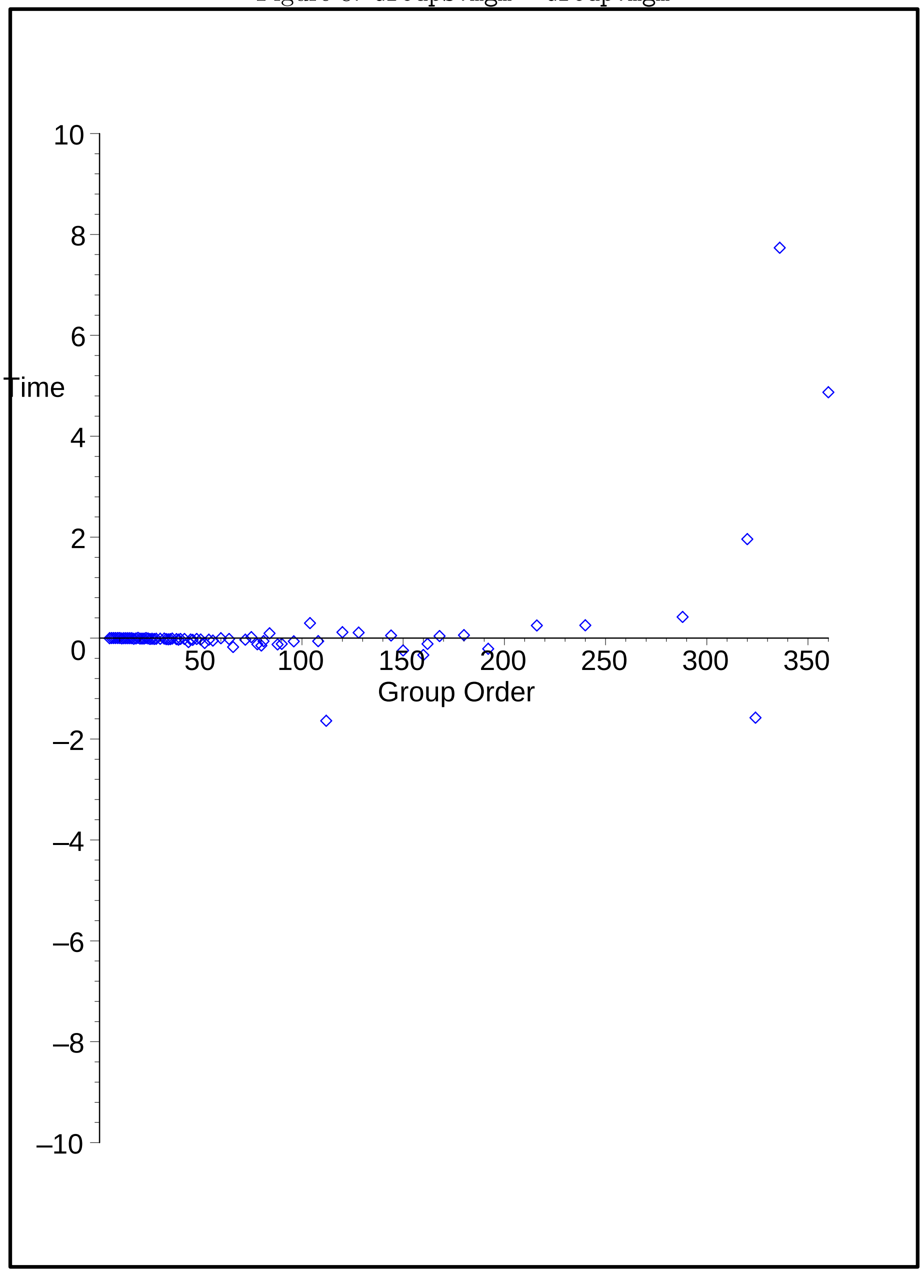


Figure 9: Comparison of `Group.mgm` and `Graph.mgm`

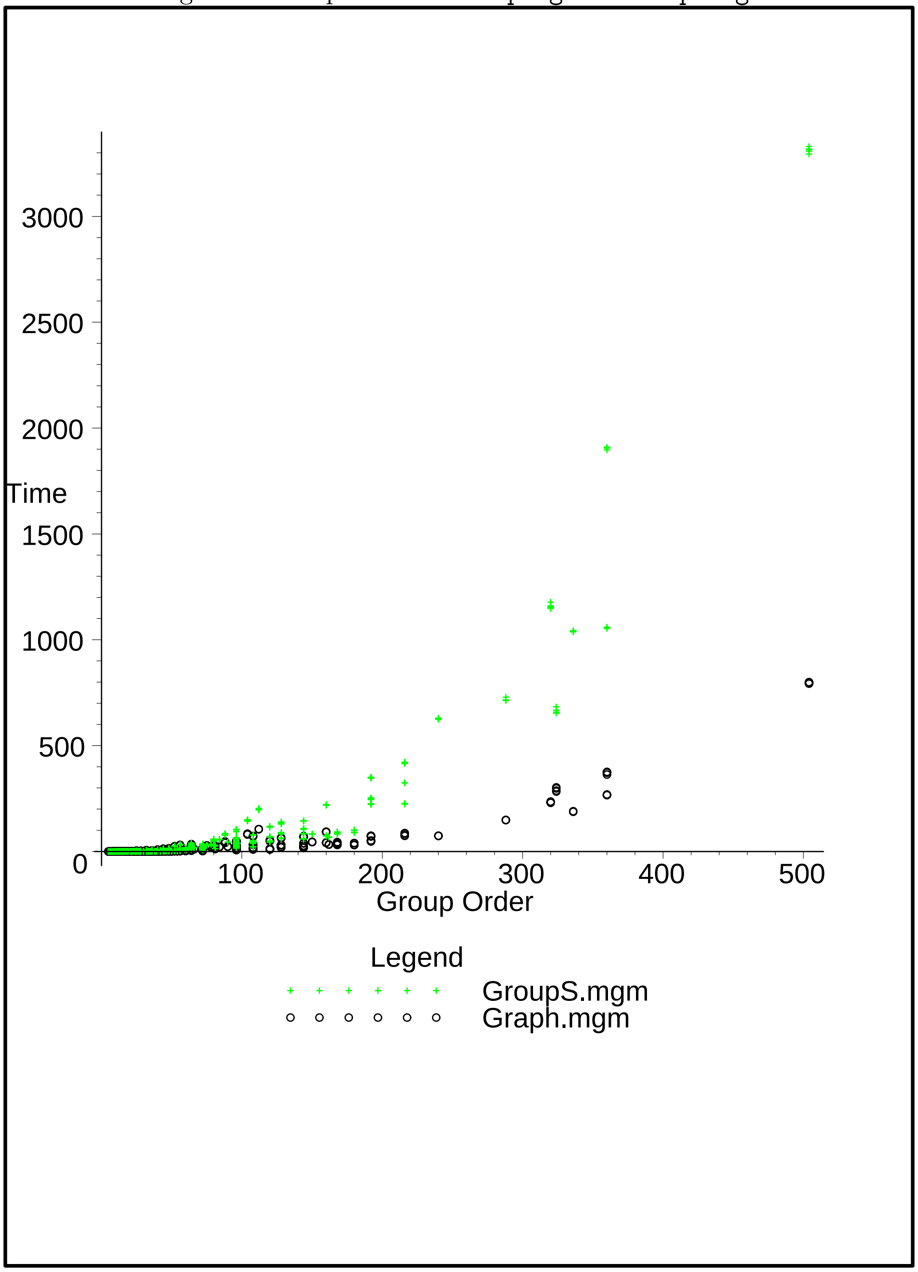


Figure 10:
$\frac{\texttt{Graph.mgm}}{\texttt{Group.mgm}}$

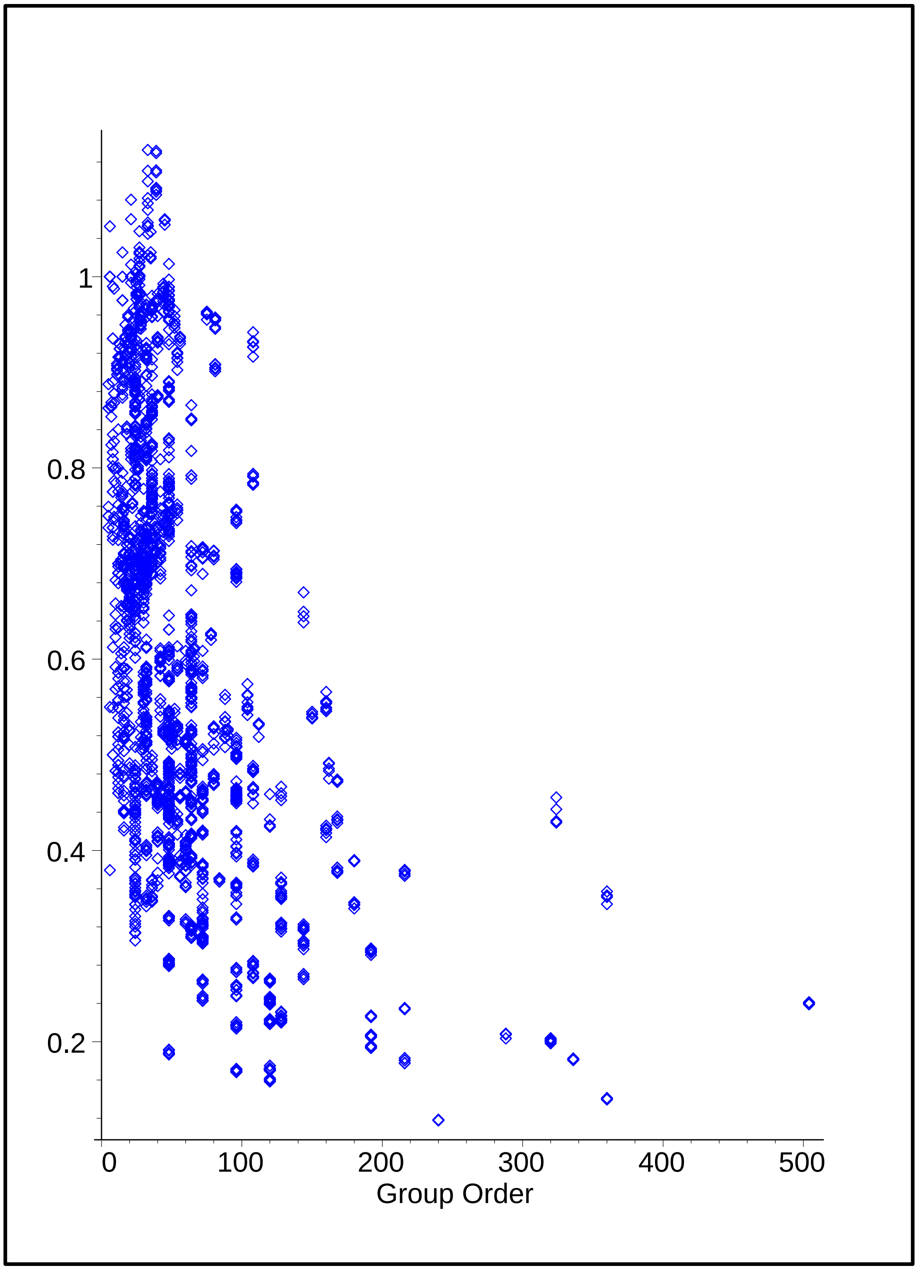
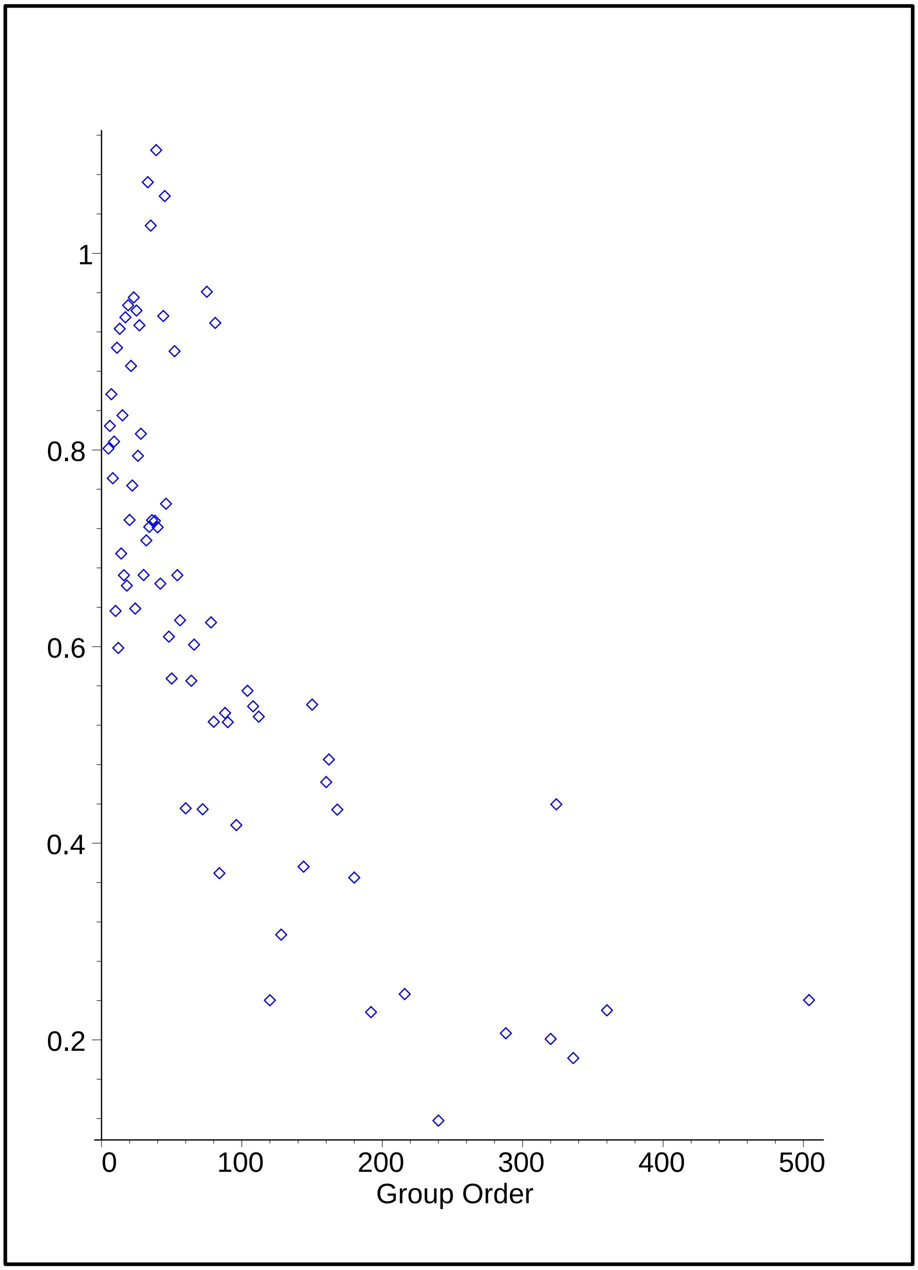


Figure 11:
$\frac{\texttt{Graph.mgm}}{\texttt{Group.mgm}}$



5.3 $PSL(2, q)$

In the analysis of the computational time data, it became apparent that the primary bottlenecks in `Group.mgm` and `GroupS.mgm` were the creation of G^* and the finding of the appropriate p, q, and r for the G^*. In order to better understand this effect the author, in coordination with Yvonne Lai [13], implemented an algorithm for determining G^* and p, q, and r for $PSL(2, q)$ groups. The outline for this algorithm came from [14] and further details were worked out by the author and Yvonne Lai with reference to [12]. In [12] all $(2, 3, 7)$-tilings were determined, and it was also determined that none of the reflections were separating. This algorithm has two primary stages, one is finding of the elements A, B, and R in $SL(2, q)$, ie the group of 2 by 2 matrices over the finite field $\mathbb{F}_q$. Where $|A| = 2$, $|B| = 3$, $|AB| = 7$, and R satisfies that

$$RAR^{-1} = A^{-1}$$
$$RBR^{-1} = B^{-1}.$$

The second stage is projecting A, B, and R into an appropriately sized symmetric group through the means of a fractional linear transform on $\mathbb{F}_q \cup \infty$.

Finding $PSL(2, q)^*$

1. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
   - It is always possible to choose a basis such that this holds since all order 2 matrices in $PSL(2, q)$ are conjugate [12].
2. Find an appropriate number of valid B's such that the AB's have different traces [14]:
   - There are 3 such B's if q is prime and not 7.
   - There is 1 such B, otherwise.
3. For each (A, B) pair, find an R satisfying
$$RAR^{-1} = A^{-1}$$
$$RBR^{-1} = B^{-1}.$$
4. If `det` R is square in $\mathbb{F}_q$, then $G^* = \langle A, B\rangle \times \mathbb{Z}_2$. [12], [14]
5. If `det` R is not square in $\mathbb{F}_q$, then $G^* = \langle A, B, R\rangle$ [12], [14].
6. Perform the fractional linear transform over $\mathbb{F}_q \cup \infty$ on A, B, and R as appropriate to embed $PSL(2, q)$ in the symmetric group [14].

   Fractional Linear Transform The Fractional Linear Transform over Q of $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is represented by the function $f(z) = \frac{az+b}{cz+d}$, where $f : Q \longrightarrow Q$.

7. Find the elements corresponding to p, q, and r satisfying

$$p = RA^{-1}$$
$$q = R$$
$$r = RB.[4]$$

After relatively few computational trials it became clear that by using a clever construction of G^* and p, q, and r such as the one for $PSL(2, q)$, the Reflective Walk Algorithm is much more efficient than the Cayley Line Graph Algorithm. However there is currently not a general efficient construction of G^* for an arbitrary tiling group.

6 Separability Results

6.1 Summary Tables

In this section we collect the separability results for all tilings of genus 2 through 13. The group data came from the work of Broughton, Dirks, Sloughter, and Vinroot [7]. The separability data was calculated using the aforementioned `Graph.mgm`, available online at [17]. Before we present the data, some notes about notation.

Summary Table Notation

p	Number of groups that separate along p.
q	Number of groups that separate along q.
r	Number of groups that separate along r.
$\{p\}$	Number of groups that separate along $\{p\}$ only.
$\{q\}$	Number of groups that separate along $\{q\}$ only.
$\{r\}$	Number of groups that separate along $\{r\}$ only.
$\{p, q\}$	Number of groups that separate along all of $\{p, q\}$ only.
$\{p, r\}$	Number of groups that separate along all of $\{p, r\}$ only.
$\{q, r\}$	Number of groups that separate along all of $\{q, r\}$ only.
$\{p, q, r\}$	Number of groups that separate along all of $\{p, q, r\}$.

Table 1: Summary Table for Genus 2 - 13

σ	p	q	r	$\{p\}$	$\{q\}$	$\{r\}$	$\{p, q\}$	$\{p, r\}$	$\{q, r\}$	$\{p, q, r\}$
2	2	2	2	0	0	2	2	0	0	0
3	3	3	2	1	1	2	2	0	0	0
4	4	6	3	0	2	3	4	0	0	0
5	2	4	1	0	2	1	2	0	0	0
6	4	6	3	0	2	3	4	0	0	0
7	2	3	1	0	1	1	2	0	0	0
8	3	4	3	0	1	3	3	0	0	0
9	5	7	3	1	3	3	4	0	0	0
10	4	4	2	1	1	2	3	0	0	0
11	2	3	1	0	1	1	2	0	0	0
12	4	6	4	0	2	4	4	0	0	0
13	2	2	1	0	0	1	2	0	0	0
Total	37	50	26	3	16	26	34	0	0	0

6.2 Separability Tables

Group Table Notation

σ is the genus of the surface.
$|G|$ is the order of the group.
G is the group used, where $G(m,n)$ is the n^{th} group of order m in the MAGMA small group database.
p, q, and r represent the reflections induced by that edge of $\triangle_0$.
Type means the type of the group:

C	Cyclic Group
A2	2-generator, Non-Cyclic Abelian Group
p-NA	Non-Abelian p-Group
S-NA-NP	Non-Abelian Solvable Group, but not a p-Group
NS	Non-Solvable Group.

Table 2: Separability of Surfaces of Genus 2

σ	$\|G\|$	(l,m,n)	Group	p	q	r	Type
2	5	$(5,5,5)$	$\mathbb{Z}_5$	-	-	-	C
2	6	$(3,6,6)$	$\mathbb{Z}_6$	Yes	Yes	-	C
2	8	$(2,8,8)$	$\mathbb{Z}_8$	-	-	-	C
2	8	$(4,4,4)$	$G(8,4)$	-	-	-	p-NA
2	10	$(2,5,10)$	$\mathbb{Z}_{10}$	-	-	-	C
2	12	$(2,6,6)$	$\mathbb{Z}_2 \times \mathbb{Z}_6$	-	-	Yes	A2
2	12	$(3,4,4)$	$G(12,1)$	-	-	Yes	S-NA-NP
2	16	$(2,4,8)$	$G(16,8)$	-	-	-	p-NA
2	24	$(2,4,6)$	$G(24,8)$	Yes	Yes	-	S-NA-NP
2	24	$(3,3,4)$	$G(24,3)$	-	-	-	S-NA-NP
2	48	$(2,3,8)$	$G(48,29)$	-	-	-	S-NA-NP

Table 3: Separability of Surfaces of Genus 3

σ	$\lvert G\rvert$	(l,m,n)	Group	p	q	r	Type
3	7	$(7,7,7)$	$\mathbb{Z}_7$	-	-	-	C
3	7	$(7,7,7)$	$\mathbb{Z}_7$	-	-	-	C
3	8	$(4,8,8)$	$\mathbb{Z}_8$	Yes	Yes	-	C
3	8	$(4,8,8)$	$\mathbb{Z}_8$	-	-	-	C
3	9	$(3,9,9)$	$\mathbb{Z}_9$	-	-	-	C
3	12	$(2,12,12)$	$\mathbb{Z}_{12}$	-	-	-	C
3	12	$(3,4,12)$	$\mathbb{Z}_{12}$	-	-	-	C
3	12	$(4,4,6)$	$G(12,1)$	-	-	-	S-NA-NP
3	14	$(2,7,14)$	$\mathbb{Z}_{14}$	-	-	-	C
3	16	$(2,8,8)$	$\mathbb{Z}_2 \times \mathbb{Z}_8$	-	-	Yes	A2
3	16	$(2,8,8)$	$G(16,6)$	-	-	-	p-NA
3	16	$(4,4,4)$	$\mathbb{Z}_4 \times \mathbb{Z}_4$	-	-	-	A2
3	16	$(4,4,4)$	$G(16,4)$	Yes	-	-	p-NA
3	24	$(3,4,12)$	$G(24,5)$	-	-	-	S-NA-NP
3	24	$(2,6,6)$	$G(24,13)$	-	-	-	S-NA-NP
3	24	$(3,3,6)$	$G(24,3)$	-	-	-	S-NA-NP
3	24	$(3,4,4)$	$G(24,12)$	-	-	Yes	S-NA-NP
3	32	$(2,4,8)$	$G(32,9)$	Yes	Yes	-	p-NA
3	32	$(2,4,8)$	$G(32,11)$	-	-	-	p-NA
3	48	$(2,3,12)$	$G(48,33)$	-	-	-	S-NA-NP
3	48	$(2,4,6)$	$G(48,48)$	-	Yes	-	S-NA-NP
3	48	$(3,3,4)$	$G(48,3)$	-	-	-	S-NA-NP
3	96	$(2,3,8)$	$G(96,64)$	-	-	-	S-NA-NP
3	168	$(2,3,7)$	$G(168,42)$	-	-	-	NS

Table 4: Separability of Surfaces of Genus 4

σ	$\|G\|$	(l, m, n)	Group	p	q	r	Type
4	9	$(9, 9, 9)$	$\mathbb{Z}_9$	-	-	-	C
4	10	$(5, 10, 10)$	$\mathbb{Z}_{10}$	-	-	-	C
4	10	$(5, 10, 10)$	$\mathbb{Z}_{10}$	Yes	Yes	-	C
4	12	$(3, 12, 12)$	$\mathbb{Z}_{12}$	-	-	-	C
4	12	$(4, 6, 12)$	$\mathbb{Z}_{12}$	-	-	-	C
4	12	$(6, 6, 6)$	$\mathbb{Z}_2 \times \mathbb{Z}_6$	-	-	-	A2
4	15	$(3, 5, 12)$	$\mathbb{Z}_{15}$	-	-	-	C
4	16	$(2, 16, 16)$	$\mathbb{Z}_{16}$	-	-	-	C
4	16	$(4, 4, 8)$	$G(16, 9)$	-	-	-	p-NA
4	18	$(2, 9, 18)$	$\mathbb{Z}_{18}$	-	-	-	C
4	18	$(3, 6, 6)$	$\mathbb{Z}_3 \times \mathbb{Z}_6$	-	-	-	A2
4	18	$(3, 6, 6)$	$G(18, 3)$	-	-	Yes	S-NA-NP
4	18	$(3, 6, 6)$	$G(18, 3)$	-	-	-	S-NA-NP
4	20	$(2, 10, 10)$	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$	-	-	Yes	A2
4	20	$(4, 4, 5)$	$G(20, 1)$	-	Yes	-	S-NA-NP
4	24	$(2, 6, 12)$	$G(24, 10)$	-	-	-	S-NA-NP
4	24	$(3, 4, 6)$	$G(24, 3)$	-	-	-	S-NA-NP
4	32	$(2, 4, 16)$	$G(32, 19)$	-	-	-	p-NA
4	36	$(2, 6, 6)$	$G(36, 10)$	Yes	Yes	-	S-NA-NP
4	36	$(2, 6, 6)$	$G(36, 12)$	-	-	-	S-NA-NP
4	36	$(3, 3, 6)$	$G(36, 11)$	-	-	-	S-NA-NP
4	36	$(3, 4, 4)$	$G(36, 9)$	Yes	Yes	-	S-NA-NP
4	40	$(2, 4, 10)$	$G(40, 8)$	Yes	Yes	-	S-NA-NP
4	60	$(2, 5, 5)$	$G(60, 5)$	-	-	-	NS
4	72	$(2, 3, 12)$	$G(72, 42)$	-	-	-	S-NA-NP
4	72	$(2, 4, 6)$	$G(72, 40)$	-	-	Yes	S-NA-NP
4	120	$(2, 4, 5)$	$G(120, 34)$	-	-	-	NS

Table 5: Separability of Surfaces of Genus 5

σ	$\|G\|$	(l, m, n)	Group	p	q	r	Type
5	11	$(11, 11, 11)$	$\mathbb{Z}_{11}$	-	-	-	C
5	11	$(11, 11, 11)$	$\mathbb{Z}_{11}$	-	-	-	C
5	12	$(6, 12, 12)$	$\mathbb{Z}_{12}$	Yes	Yes	-	C
5	15	$(3, 15, 15)$	$\mathbb{Z}_{15}$	-	-	-	C
5	16	$(4, 8, 8)$	$\mathbb{Z}_2 \times \mathbb{Z}_8$	-	-	-	A2
5	16	$(4, 8, 8)$	$G(16, 6)$	-	-	-	p-NA
5	20	$(2, 20, 20)$	$\mathbb{Z}_{20}$	-	-	-	C
5	20	$(4, 4, 10)$	$G(20, 1)$	-	-	-	S-NA-NP
5	22	$(2, 11, 22)$	$\mathbb{Z}_{22}$	-	-	-	C
5	24	$(2, 12, 12)$	$\mathbb{Z}_2 \times \mathbb{Z}_{12}$	-	-	Yes	A2
5	24	$(3, 6, 6)$	$G(24, 13)$	-	-	-	S-NA-NP
5	24	$(4, 4, 6)$	$G(24, 7)$	-	Yes	-	S-NA-NP
5	30	$(2, 6, 15)$	$G(30, 2)$	-	-	-	S-NA-NP
5	32	$(2, 8, 8)$	$G(32, 5)$	-	-	-	p-NA
5	32	$(2, 8, 8)$	$G(32, 7)$	-	-	-	p-NA
5	32	$(4, 4, 4)$	$G(32, 2)$	-	-	-	p-NA
5	32	$(4, 4, 4)$	$G(32, 6)$	-	-	-	p-NA
5	40	$(2, 4, 20)$	$G(40, 5)$	-	-	-	S-NA-NP
5	48	$(2, 4, 12)$	$G(48, 14)$	Yes	Yes	-	S-NA-NP
5	48	$(3, 4, 4)$	$G(48, 30)$	-	-	-	S-NA-NP
5	48	$(3, 4, 4)$	$G(48, 30)$	-	-	-	S-NA-NP
5	60	$(3, 3, 5)$	$G(60, 5)$	-	-	-	NS
5	64	$(2, 4, 8)$	$G(64, 8)$	-	-	-	p-NA
5	64	$(2, 4, 8)$	$G(64, 32)$	-	-	-	p-NA
5	80	$(2, 5, 5)$	$G(80, 49)$	-	-	-	S-NA-NP
5	96	$(2, 4, 6)$	$G(96, 195)$	-	-	-	S-NA-NP
5	96	$(3, 3, 4)$	$G(96, 3)$	-	-	-	S-NA-NP
5	120	$(2, 3, 10)$	$G(120, 35)$	-	-	-	NS
5	160	$(2, 4, 5)$	$G(160, 234)$	-	Yes	-	S-NA-NP
5	192	$(2, 3, 8)$	$G(192, 181)$	-	-	-	S-NA-NP

Table 6: Separability of Surfaces of Genus 6

σ	$\lvert G\rvert$	(l,m,n)	Group	p	q	r	Type
6	13	$(13,13,13)$	$\mathbb{Z}_{13}$	-	-	-	C
6	13	$(13,13,13)$	$\mathbb{Z}_{13}$	-	-	-	C
6	13	$(13,13,13)$	$\mathbb{Z}_{13}$	-	-	-	C
6	14	$(7,14,14)$	$\mathbb{Z}_{14}$	-	-	-	C
6	14	$(7,14,14)$	$\mathbb{Z}_{14}$	-	-	-	C
6	14	$(7,14,14)$	$\mathbb{Z}_{14}$	Yes	Yes	-	C
6	15	$(5,15,15)$	$\mathbb{Z}_{15}$	-	-	-	C
6	15	$(5,15,15)$	$\mathbb{Z}_{15}$	-	-	-	C
6	16	$(4,16,16)$	$\mathbb{Z}_{16}$	-	-	-	C
6	18	$(3,18,18)$	$\mathbb{Z}_{18}$	-	-	-	C
6	20	$(4,5,20)$	$\mathbb{Z}_{20}$	-	-	-	C
6	21	$(3,7,21)$	$\mathbb{Z}_{21}$	-	-	-	C
6	24	$(2,24,24)$	$\mathbb{Z}_{24}$	-	-	-	C
6	24	$(3,8,8)$	$G(24,1)$	-	-	Yes	S-NA-NP
6	24	$(4,4,12)$	$G(24,4)$	-	-	-	S-NA-NP
6	24	$(4,6,6)$	$G(24,3)$	-	-	-	S-NA-NP
6	24	$(4,6,6)$	$G(24,10)$	-	-	Yes	S-NA-NP
6	25	$(2,5,5)$	$\mathbb{Z}_5 \times \mathbb{Z}_5$	-	-	-	A2
6	26	$(2,13,26)$	$\mathbb{Z}_{26}$	-	-	-	C
6	28	$(2,14,14)$	$\mathbb{Z}_2 \times \mathbb{Z}_{14}$	-	-	Yes	A2
6	28	$(4,4,7)$	$G(28,1)$	-	Yes	-	S-NA-NP
6	30	$(2,10,15)$	$G(30,1)$	-	-	-	S-NA-NP
6	36	$(2,9,9)$	$G(36,3)$	-	-	-	S-NA-NP
6	48	$(2,4,24)$	$G(48,6)$	-	-	-	S-NA-NP
6	48	$(2,6,8)$	$G(48,15)$	Yes	Yes	-	S-NA-NP
6	48	$(2,6,8)$	$G(48,29)$	-	-	-	S-NA-NP
6	50	$(2,5,10)$	$G(50,3)$	-	-	-	S-NA-NP
6	56	$(2,4,14)$	$G(56,7)$	Yes	Yes	-	S-NA-NP
6	72	$(2,4,9)$	$G(72,15)$	-	Yes	-	S-NA-NP
6	75	$(3,3,5)$	$G(75,2)$	-	-	-	S-NA-NP
6	120	$(2,4,6)$	$G(120,34)$	Yes	Yes	-	NS
6	150	$(2,3,10)$	$G(150,5)$	-	-	-	S-NA-NP

Table 7: Separability of Surfaces of Genus 7

σ	$\lvert G\rvert$	(l,m,n)	Group	p	q	r	Type
7	15	$(15,15,15)$	$\mathbb{Z}_{15}$	-	-	-	C
7	16	$(8,16,16)$	$\mathbb{Z}_{16}$	-	-	-	C
7	16	$(8,16,16)$	$\mathbb{Z}_{16}$	-	-	-	C
7	16	$(8,16,16)$	$\mathbb{Z}_{16}$	Yes	Yes	-	C
7	18	$(6,9,18)$	$\mathbb{Z}_{18}$	-	-	-	C
7	18	$(6,9,18)$	$\mathbb{Z}_{18}$	-	-	-	C
7	20	$(4,10,20)$	$\mathbb{Z}_{20}$	-	-	-	C
7	21	$(3,21,21)$	$\mathbb{Z}_{21}$	-	-	-	C
7	24	$(3,8,24)$	$\mathbb{Z}_{24}$	-	-	-	C
7	24	$(4,6,12)$	$\mathbb{Z}_2 \times \mathbb{Z}_{12}$	-	-	-	A2
7	24	$(6,6,6)$	$G(24,3)$	-	-	-	S-NA-NP
7	27	$(3,9,9)$	$\mathbb{Z}_3 \times \mathbb{Z}_9$	-	-	-	A2
7	28	$(2,28,28)$	$\mathbb{Z}_{28}$	-	-	-	C
7	28	$(4,4,14)$	$G(28,1)$	-	-	-	S-NA-NP
7	30	$(2,15,30)$	$\mathbb{Z}_{30}$	-	-	-	C
7	32	$(2,16,16)$	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$	-	-	Yes	A2
7	32	$(2,16,16)$	$G(32,17)$	-	-	-	p-NA
7	32	$(4,4,8)$	$G(32,10)$	-	-	-	p-NA
7	32	$(4,4,8)$	$G(32,11)$	-	-	-	p-NA
7	32	$(4,4,8)$	$G(32,13)$	-	-	-	p-NA
7	32	$(4,4,8)$	$G(32,14)$	-	Yes	-	p-NA
7	36	$(3,4,12)$	$G(36,6)$	-	-	-	S-NA-NP
7	42	$(2,6,21)$	$G(42,4)$	-	-	-	S-NA-NP
7	48	$(2,6,12)$	$G(48,33)$	-	-	-	S-NA-NP
7	48	$(2,4,6)$	$G(48,32)$	-	-	-	S-NA-NP
7	54	$(2,6,9)$	$G(54,3)$	-	-	-	S-NA-NP
7	56	$(2,4,28)$	$G(56,4)$	-	-	-	S-NA-NP
7	64	$(2,4,16)$	$G(64,38)$	Yes	Yes	-	p-NA
7	64	$(2,4,16)$	$G(64,41)$	-	-	-	p-NA
7	72	$(3,3,6)$	$G(72,25)$	-	-	-	S-NA-NP
7	144	$(2,3,12)$	$G(144,127)$	-	-	-	S-NA-NP
7	504	$(2,3,7)$	$G(504,156)$	-	-	-	NS

Table 8: Separability of Surfaces of Genus 8

σ	$\|G\|$	(l,m,n)	Group	p	q	r	Type
8	17	$(17,17,17)$	$\mathbb{Z}_{17}$	-	-	-	C
8	17	$(17,17,17)$	$\mathbb{Z}_{17}$	-	-	-	C
8	17	$(17,17,17)$	$\mathbb{Z}_{17}$	-	-	-	C
8	18	$(9,18,18)$	$\mathbb{Z}_{18}$	Yes	Yes	-	C
8	18	$(9,18,18)$	$\mathbb{Z}_{18}$	-	-	-	C
8	20	$(10,10,10)$	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$	-	-	-	A2
8	20	$(5,20,20)$	$\mathbb{Z}_{20}$	-	-	-	C
8	20	$(5,20,20)$	$\mathbb{Z}_{20}$	-	-	-	C
8	24	$(3,24,24)$	$\mathbb{Z}_{24}$	-	-	-	C
8	24	$(4,12,12)$	$G(24,11)$	-	-	-	S-NA-NP
8	24	$(6,6,12)$	$G(24,10)$	-	-	-	S-NA-NP
8	24	$(6,8,8)$	$G(24,1)$	-	-	-	S-NA-NP
8	30	$(3,10,10)$	$G(30,1)$	-	-	Yes	S-NA-NP
8	30	$(5,6,6)$	$G(30,2)$	-	-	Yes	S-NA-NP
8	32	$(2,32,32)$	$\mathbb{Z}_{32}$	-	-	-	C
8	32	$(4,4,16)$	$G(32,20)$	-	-	-	p-NA
8	34	$(2,17,34)$	$\mathbb{Z}_{34}$	-	-	-	C
8	36	$(2,18,18)$	$\mathbb{Z}_2 \times \mathbb{Z}_{18}$	-	-	Yes	A2
8	36	$(4,4,9)$	$G(36,1)$	-	Yes	-	S-NA-NP
8	40	$(2,10,20)$	$G(40,10)$	-	-	-	S-NA-NP
8	48	$(2,6,24)$	$G(48,25)$	-	-	-	S-NA-NP
8	48	$(2,8,12)$	$G(48,17)$	-	-	-	S-NA-NP
8	48	$(3,4,8)$	$G(48,28)$	-	-	-	S-NA-NP
8	60	$(2,6,10)$	$G(60,8)$	Yes	Yes	-	S-NA-NP
8	64	$(2,4,32)$	$G(64,53)$	-	-	-	p-NA
8	72	$(2,4,18)$	$G(72,8)$	Yes	Yes	-	S-NA-NP
8	168	$(3,3,4)$	$G(168,42)$	-	-	-	NS
8	168	$(3,3,4)$	$G(168,42)$	-	-	-	NS
8	336	$(2,3,8)$	$G(336,208)$	-	-	-	NS
8	336	$(2,3,8)$	$G(336,208)$	-	-	-	NS

Table 9: Separability of Surfaces of Genus 9

σ	$\|G\|$	(l,m,n)	Group	p	q	r	Type
9	19	$(19,19,19)$	$\mathbb{Z}_{19}$	-	-	-	C
9	19	$(19,19,19)$	$\mathbb{Z}_{19}$	-	-	-	C
9	19	$(19,19,19)$	$\mathbb{Z}_{19}$	-	-	-	C
9	19	$(19,19,19)$	$\mathbb{Z}_{19}$	-	-	-	C
9	20	$(10,20,20)$	$\mathbb{Z}_{20}$	-	-	-	C
9	20	$(10,20,20)$	$\mathbb{Z}_{20}$	Yes	Yes	-	C
9	21	$(7,21,21)$	$\mathbb{Z}_{21}$	-	-	-	C
9	21	$(7,21,21)$	$\mathbb{Z}_{21}$	-	-	-	C
9	21	$(7,21,21)$	$\mathbb{Z}_{21}$	-	-	-	C
9	24	$(4,24,24)$	$\mathbb{Z}_{24}$	-	-	-	C
9	24	$(4,24,24)$	$\mathbb{Z}_{24}$	-	-	-	C
9	24	$(6,12,12)$	$\mathbb{Z}_2 \times \mathbb{Z}_{12}$	-	-	-	A2
9	24	$(6,8,24)$	$\mathbb{Z}_{24}$	-	-	-	C
9	24	$(8,8,12)$	$G(24,1)$	-	-	-	S-NA-NP
9	24	$(8,8,12)$	$G(24,1)$	-	-	-	S-NA-NP
9	27	$(3,27,27)$	$\mathbb{Z}_{27}$	-	-	-	C
9	28	$(4,7,28)$	$\mathbb{Z}_{28}$	-	-	-	C
9	30	$(3,10,30)$	$\mathbb{Z}_{30}$	-	-	-	C
9	32	$(4,8,8)$	$\mathbb{Z}_4 \times \mathbb{Z}_8$	-	-	-	A2
9	32	$(4,8,8)$	$G(32,4)$	-	-	-	p-NA
9	32	$(4,8,8)$	$G(32,5)$	-	-	-	p-NA
9	32	$(4,8,8)$	$G(32,8)$	-	-	-	p-NA
9	32	$(4,8,8)$	$G(32,12)$	-	-	-	p-NA
9	32	$(4,8,8)$	$G(32,12)$	-	-	Yes	p-NA
9	36	$(2,36,36)$	$\mathbb{Z}_{36}$	-	-	-	C
9	36	$(4,4,18)$	$G(36,1)$	-	-	-	S-NA-NP
9	38	$(2,19,38)$	$\mathbb{Z}_{38}$	-	-	-	C
9	40	$(2,20,20)$	$\mathbb{Z}_2 \times \mathbb{Z}_{20}$	-	-	Yes	A2
9	40	$(4,4,10)$	$G(40,7)$	-	Yes	-	S-NA-NP
9	42	$(2,14,21)$	$G(42,3)$	-	-	-	S-NA-NP
9	48	$(2,12,12)$	$G(48,21)$	-	-	-	S-NA-NP
9	48	$(2,12,12)$	$G(48,31)$	-	-	-	S-NA-NP
9	48	$(2,12,12)$	$G(48,31)$	-	-	-	S-NA-NP
9	48	$(2,8,24)$	$G(48,4)$	-	-	-	S-NA-NP
9	48	$(2,8,24)$	$G(48,5)$	-	-	-	S-NA-NP
9	48	$(3,4,12)$	$G(48,31)$	-	-	-	S-NA-NP
9	48	$(3,6,6)$	$G(48,32)$	-	-	-	S-NA-NP
9	48	$(4,4,6)$	$G(48,19)$	-	-	-	S-NA-NP
9	48	$(4,4,6)$	$G(48,30)$	-	-	-	S-NA-NP
9	48	$(4,4,6)$	$G(48,30)$	-	-	-	S-NA-NP

Table 10: Separability of Surfaces of Genus 9 (Cont.)

σ	$\lvert G\rvert$	(l,m,n)	Group	p	q	r	Type
9	60	$(3,5,5)$	$G(60,5)$	-	-	-	NS
9	60	$(3,5,5)$	$G(60,5)$	-	-	-	NS
9	64	$(2,8,8)$	$G(64,4)$	-	-	-	p-NA
9	64	$(2,8,8)$	$G(64,6)$	-	-	-	p-NA
9	64	$(2,8,8)$	$G(64,10)$	-	-	-	p-NA
9	64	$(2,8,8)$	$G(64,12)$	Yes	Yes	-	p-NA
9	64	$(2,8,8)$	$G(64,36)$	-	-	-	p-NA
9	64	$(4,4,4)$	$G(64,23)$	Yes	-	-	p-NA
9	64	$(4,4,4)$	$G(64,34)$	Yes	Yes	-	p-NA
9	64	$(4,4,4)$	$G(64,35)$	-	-	-	p-NA
9	64	$(4,4,4)$	$G(64,35)$	-	-	-	p-NA
9	72	$(2,4,36)$	$G(72,5)$	-	-	-	S-NA-NP
9	80	$(2,4,20)$	$G(80,14)$	Yes	Yes	-	S-NA-NP
9	96	$(2,4,12)$	$G(96,13)$	-	-	-	S-NA-NP
9	96	$(2,4,12)$	$G(96,186)$	-	-	-	S-NA-NP
9	96	$(2,4,12)$	$G(96,187)$	-	Yes	-	S-NA-NP
9	96	$(2,6,6)$	$G(96,70)$	-	-	-	S-NA-NP
9	96	$(3,3,6)$	$G(96,3)$	-	-	-	S-NA-NP
9	96	$(3,4,4)$	$G(96,67)$	-	-	-	S-NA-NP
9	96	$(3,4,4)$	$G(96,227)$	-	-	-	S-NA-NP
9	120	$(2,5,6)$	$G(120,34)$	-	-	-	NS
9	120	$(2,5,6)$	$G(120,35)$	-	-	-	NS
9	128	$(2,4,8)$	$G(128,75)$	-	Yes	-	p-NA
9	128	$(2,4,8)$	$G(128,134)$	-	-	Yes	p-NA
9	128	$(2,4,8)$	$G(128,136)$	-	-	-	p-NA
9	128	$(2,4,8)$	$G(128,138)$	-	-	-	p-NA
9	160	$(2,5,5)$	$G(160,199)$	-	-	-	S-NA-NP
9	192	$(2,3,12)$	$G(192,194)$	-	-	-	S-NA-NP
9	192	$(2,4,6)$	$G(192,955)$	-	-	-	S-NA-NP
9	192	$(2,4,6)$	$G(192,990)$	-	-	-	S-NA-NP
9	320	$(2,4,5)$	$G(320,1582)$	-	-	-	S-NA-NP

Table 11: Separability of Surfaces of Genus 10

σ	$\|G\|$	(l,m,n)	Group	p	q	r	Type
10	21	$(21,21,21)$	$\mathbb{Z}_{21}$	-	-	-	C
10	21	$(21,21,21)$	$\mathbb{Z}_{21}$	-	-	-	C
10	22	$(11,22,22)$	$\mathbb{Z}_{22}$	-	-	-	C
10	22	$(11,22,22)$	$\mathbb{Z}_{22}$	-	-	-	C
10	22	$(11,22,22)$	$\mathbb{Z}_{22}$	-	-	-	C
10	22	$(11,22,22)$	$\mathbb{Z}_{22}$	-	-	-	C
10	22	$(11,22,22)$	$\mathbb{Z}_{22}$	Yes	Yes	-	C
10	24	$(12,12,120$	$G(24,11)$	-	-	-	S-NA-NP
10	24	$(6,24,24)$	$\mathbb{Z}_{24}$	-	-	-	C
10	24	$(8,12,24)$	$\mathbb{Z}_{24}$	-	-	-	C
10	24	$(8,12,24)$	$\mathbb{Z}_{24}$	-	-	-	C
10	25	$(5,25,25)$	$\mathbb{Z}_{25}$	-	-	-	C
10	25	$(5,25,25)$	$\mathbb{Z}_{25}$	-	-	-	C
10	27	$(9,9,9)$	$\mathbb{Z}_3 \times \mathbb{Z}_9$	-	-	-	A2
10	28	$(4,14,28)$	$\mathbb{Z}_{28}$	-	-	-	C
10	30	$(3,30,30)$	$\mathbb{Z}_{30}$	-	-	-	C
10	30	$(5,6,30)$	$\mathbb{Z}_{30}$	-	-	-	C
10	30	$(6,6,15)$	$G(30,2)$	-	-	-	S-NA-NP
10	33	$(3,11,33)$	$\mathbb{Z}_{33}$	-	-	-	C
10	36	$(3,12,12)$	$\mathbb{Z}_3 \times \mathbb{Z}_{12}$	-	-	-	A2
10	36	$(3,12,12)$	$G(36,6)$	-	-	Yes	S-NA-NP
10	36	$(3,12,12)$	$G(36,6)$	-	-	-	S-NA-NP
10	36	$(4,6,12)$	$G(36,6)$	-	-	-	S-NA-NP
10	36	$(6,6,6)$	$\mathbb{Z}_6 \times \mathbb{Z}_6$	-	-	-	A2
10	36	$(6,6,6)$	$G(36,12)$	-	-	-	S-NA-NP
10	36	$(6,6,6)$	$G(36,12)$	Yes	-	-	S-NA-NP
10	40	$(2,40,40)$	$\mathbb{Z}_{40}$	-	-	-	C
10	40	$(4,4,20)$	$G(40,4)$	-	-	-	S-NA-NP
10	42	$(2,21,42)$	$\mathbb{Z}_{42}$	-	-	-	C
10	44	$(2,22,22)$	$\mathbb{Z}_2 \times \mathbb{Z}_{22}$	-	-	Yes	A2
10	44	$(4,4,11)$	$G(44,1)$	-	Yes	-	S-NA-NP
10	48	$(2,12,24)$	$G(48,26)$	-	-	-	S-NA-NP
10	54	$(2,9,18)$	$G(54,4)$	-	-	-	S-NA-NP
10	54	$(3,6,6)$	$G(54,5)$	-	-	-	S-NA-NP
10	54	$(3,6,6)$	$G(54,5)$	-	-	-	S-NA-NP
10	54	$(3,6,6)$	$G(54,10)$	-	-	-	S-NA-NP
10	54	$(3,6,6)$	$G(54,12)$	-	-	-	S-NA-NP
10	60	$(3,6,30)$	$G(60,10)$	-	-	-	S-NA-NP
10	72	$(2,6,12)$	$G(72,23)$	Yes	Yes	-	S-NA-NP
10	72	$(2,6,12)$	$G(72,28)$	-	-	-	S-NA-NP

Table 12: Separability of Surfaces of Genus 10 (Cont.)

σ	$\|G\|$	(l, m, n)	Group	p	q	r	Type
10	72	$(2, 6, 12)$	$G(72, 30)$	-	-	-	S-NA-NP
10	72	$(3, 3, 12)$	$G(72, 25)$	-	-	-	S-NA-NP
10	72	$(3, 4, 6)$	$G(72, 42)$	-	-	-	S-NA-NP
10	80	$(2, 4, 40)$	$G(80, 6)$	-	-	-	S-NA-NP
10	81	$(3, 3, 9)$	$G(81, 7)$	-	-	-	p-NA
10	81	$(3, 3, 9)$	$G(81, 9)$	-	-	-	p-NA
10	88	$(2, 4, 22)$	$G(88, 7)$	Yes	Yes	-	S-NA-NP
10	108	$(2, 4, 12)$	$G(108, 15)$	-	-	-	S-NA-NP
10	108	$(2, 6, 6)$	$G(108, 17)$	-	-	-	S-NA-NP
10	108	$(2, 6, 6)$	$G(108, 25)$	-	-	-	S-NA-NP
10	108	$(2, 6, 6)$	$G(108, 38)$	-	-	-	S-NA-NP
10	108	$(3, 3, 6)$	$G(108, 22)$	-	-	-	S-NA-NP
10	108	$(3, 4, 4)$	$G(108, 15)$	-	-	-	S-NA-NP
10	108	$(3, 4, 4)$	$G(108, 37)$	-	-	-	S-NA-NP
10	144	$(2, 3, 24)$	$G(144, 122)$	-	-	-	S-NA-NP
10	162	$(2, 3, 8)$	$G(162, 14)$	-	-	-	S-NA-NP
10	168	$(2, 4, 7)$	$G(168, 42)$	-	-	-	NS
10	180	$(2, 3, 15)$	$G(180, 19)$	-	-	-	NS
10	216	$(2, 3, 12)$	$G(216, 92)$	-	-	-	S-NA-NP
10	216	$(2, 4, 6)$	$G(216, 87)$	-	-	-	S-NA-NP
10	216	$(2, 4, 6)$	$G(216, 158)$	-	-	-	S-NA-NP
10	216	$(3, 3, 4)$	$G(216, 153)$	-	-	-	S-NA-NP
10	324	$(2, 3, 9)$	$G(324, 160)$	-	-	-	S-NA-NP
10	360	$(2, 4, 5)$	$G(360, 118)$	-	-	-	NS

Table 13: Separability of Surfaces of Genus 11

σ	$\|G\|$	(l,m,n)	Group	p	q	r	Type
11	23	$(23,23,23)$	$\mathbb{Z}_{23}$	-	-	-	C
11	23	$(23,23,23)$	$\mathbb{Z}_{23}$	-	-	-	C
11	23	$(23,23,23)$	$\mathbb{Z}_{23}$	-	-	-	C
11	23	$(23,23,23)$	$\mathbb{Z}_{23}$	-	-	-	C
11	24	$(12,24,24)$	$\mathbb{Z}_{24}$	-	-	-	C
11	24	$(12,24,24)$	$\mathbb{Z}_{24}$	Yes	Yes	-	C
11	30	$(6,10,15)$	$\mathbb{Z}_{30}$	-	-	-	C
11	32	$(4,16,16)$	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$	-	-	-	A2
11	32	$(4,16,16)$	$G(32,17)$	-	-	-	p-NA
11	32	$(8,8,8)$	$G(32,17)$	-	-	-	p-NA
11	32	$(8,8,8)$	$G(32,17)$	-	-	-	p-NA
11	33	$(3,33,33)$	$\mathbb{Z}_{33}$	-	-	-	C
11	44	$(2,44,44)$	$\mathbb{Z}_{44}$	-	-	-	C
11	44	$(4,4,22)$	$G(44,1)$	-	-	-	S-NA-NP
11	46	$(2,23,46)$	$\mathbb{Z}_{46}$	-	-	-	C
11	48	$(2,24,24)$	$\mathbb{Z}_2 \times \mathbb{Z}_{24}$	-	-	Yes	A2
11	48	$(2,24,24)$	$G(48,24)$	-	-	-	S-NA-NP
11	48	$(3,8,8)$	$G(48,28)$	-	-	-	S-NA-NP
11	48	$(3,8,8)$	$G(48,29)$	-	-	-	S-NA-NP
11	48	$(4,4,12)$	$G(48,11)$	-	-	-	S-NA-NP
11	48	$(4,4,12)$	$G(48,12)$	-	-	-	S-NA-NP
11	48	$(4,4,12)$	$G(48,13)$	-	Yes	-	S-NA-NP
11	48	$(4,6,6)$	$G(48,32)$	-	-	-	S-NA-NP
11	48	$(4,6,6)$	$G(48,32)$	-	-	-	S-NA-NP
11	48	$(4,6,6)$	$G(48,32)$	-	-	-	S-NA-NP
11	64	$(2,8,16)$	$G(64,40)$	-	-	-	p-NA
11	64	$(2,8,16)$	$G(64,42)$	-	-	-	p-NA
11	66	$(2,6,33)$	$G(66,2)$	-	-	-	S-NA-NP
11	88	$(2,4,44)$	$G(88,4)$	-	-	-	S-NA-NP
11	96	$(2,4,24)$	$G(96,28)$	Yes	Yes	-	S-NA-NP
11	96	$(2,4,24)$	$G(96,32)$	-	-	-	S-NA-NP
11	96	$(2,6,8)$	$G(96,189)$	-	-	-	S-NA-NP
11	96	$(2,6,8)$	$G(96,190)$	-	-	-	S-NA-NP
11	120	$(2,6,6)$	$G(120,34)$	-	-	-	NS
11	120	$(3,4,4)$	$G(120,34)$	-	-	-	NS
11	240	$(2,4,6)$	$G(240,189)$	-	-	-	S-NA-NP

Table 14: Separability of Surfaces of Genus 12

σ	$\|G\|$	(l,m,n)	Group	p	q	r	Type
12	25	$(25,25,25)$	$\mathbb{Z}_{25}$	-	-	-	C
12	25	$(25,25,25)$	$\mathbb{Z}_{25}$	-	-	-	C
12	25	$(25,25,25)$	$\mathbb{Z}_{25}$	-	-	-	C
12	26	$(13,26,26)$	$\mathbb{Z}_{26}$	-	-	-	C
12	26	$(13,26,26)$	$\mathbb{Z}_{26}$	-	-	-	C
12	26	$(13,26,26)$	$\mathbb{Z}_{26}$	-	-	-	C
12	26	$(13,26,26)$	$\mathbb{Z}_{26}$	-	-	-	C
12	26	$(13,26,26)$	$\mathbb{Z}_{26}$	-	-	-	C
12	26	$(13,26,26)$	$\mathbb{Z}_{26}$	Yes	Yes	-	C
12	27	$(9,27,27)$	$\mathbb{Z}_{27}$	-	-	-	C
12	27	$(9,27,27)$	$\mathbb{Z}_{27}$	-	-	-	C
12	27	$(9,27,27)$	$\mathbb{Z}_{27}$	-	-	-	C
12	28	$(14,15,15)$	$\mathbb{Z}_2 \times \mathbb{Z}_{14}$	-	-	-	A2
12	28	$(14,14,14)$	$\mathbb{Z}_2 \times \mathbb{Z}_{14}$	-	-	-	A2
12	28	$(7,28,28)$	$\mathbb{Z}_{28}$	-	-	-	C
12	28	$(7,28,28)$	$\mathbb{Z}_{28}$	-	-	-	C
12	28	$(7,28,28)$	$\mathbb{Z}_{28}$	-	-	-	C
12	30	$(10,10,15)$	$G(30,1)$	-	-	-	S-NA-NP
12	30	$(10,10,15)$	$G(30,1)$	-	-	-	S-NA-NP
12	30	$(5,30,30)$	$\mathbb{Z}_{30}$	-	-	-	C
12	30	$(5,30,30)$	$\mathbb{Z}_{30}$	-	-	-	C
12	30	$(6,15,30)$	$\mathbb{Z}_{30}$	-	-	-	C
12	32	$(4,32,32)$	$\mathbb{Z}_{32}$	-	-	-	C
12	35	$(5,7,35)$	$\mathbb{Z}_{35}$	-	-	-	C
12	36	$(3,36,36)$	$\mathbb{Z}_{36}$	-	-	-	C
12	36	$(4,9,36)$	$\mathbb{Z}_{36}$	-	-	-	C
12	36	$(6,9,9)$	$G(36,3)$	-	-	-	S-NA-NP
12	39	$(3,13,39)$	$\mathbb{Z}_{39}$	-	-	-	C

Table 15: Separability of Surfaces of Genus 12 (Cont.)

σ	$\|G\|$	(l, m, n)	Group	p	q	r	Type
12	40	$(4, 10, 10)$	$G(40, 10)$	-	-	Yes	S-NA-NP
12	40	$(5, 8, 8)$	$G(40, 1)$	-	-	Yes	S-NA-NP
12	42	$(3, 14, 14)$	$G(42, 3)$	-	-	Yes	S-NA-NP
12	42	$(6, 6, 7)$	$G(42, 4)$	-	Yes	-	S-NA-NP
12	48	$(2, 48, 48)$	$\mathbb{Z}_{48}$	-	-	-	C
12	48	$(4, 4, 24)$	$G(48, 8)$	-	-	-	S-NA-NP
12	48	$(4, 6, 8)$	$G(48, 16)$	-	-	-	S-NA-NP
12	48	$(4, 6, 8)$	$G(48, 28)$	-	-	-	S-NA-NP
12	50	$(2, 25, 50)$	$\mathbb{Z}_{50}$	-	-	-	C
12	52	$(2, 26, 26)$	$\mathbb{Z}_2 \times \mathbb{Z}_{26}$	-	-	Yes	A2
12	52	$(4, 4, 13)$	$G(52, 1)$	-	Yes	-	S-NA-NP
12	56	$(2, 14, 28)$	$G(56, 9)$	-	-	-	S-NA-NP
12	60	$(2, 10, 30)$	$G(60, 11)$	-	-	-	S-NA-NP
12	60	$(2, 15, 15)$	$G(60, 9)$	-	-	-	S-NA-NP
12	80	$(2, 8, 10)$	$G(80, 15)$	Yes	Yes	-	S-NA-NP
12	84	$(2, 6, 14)$	$G(84, 8)$	Yes	Yes	-	S-NA-NP
12	96	$(2, 4, 48)$	$G(96, 7)$	-	-	-	S-NA-NP
12	104	$(2, 4, 26)$	$G(104, 8)$	Yes	Yes	-	S-NA-NP
12	120	$(2, 4, 15)$	$G(120, 138)$	-	-	-	S-NA-NP

Table 16: Separability of Surfaces of Genus 13

σ	$\|G\|$	(l,m,n)	Group	p	q	r	Type
13	27	$(27,27,27)$	$\mathbb{Z}_{27}$	-	-	-	C
13	27	$(27,27,27)$	$\mathbb{Z}_{27}$	-	-	-	C
13	27	$(27,27,27)$	$\mathbb{Z}_{27}$	-	-	-	C
13	28	$(14,28,28)$	$\mathbb{Z}_{28}$	-	-	-	C
13	28	$(14,28,28)$	$\mathbb{Z}_{28}$	-	-	-	C
13	28	$(14,28,28)$	$\mathbb{Z}_{28}$	Yes	Yes	-	C
13	30	$(10,15,30)$	$\mathbb{Z}_{30}$	-	-	-	C
13	30	$(10,15,30)$	$\mathbb{Z}_{30}$	-	-	-	C
13	30	$(10,15,30)$	$\mathbb{Z}_{30}$	-	-	-	C
13	32	$(8,16,16)$	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$	-	-	-	A2
13	32	$(8,16,16)$	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$	-	-	-	A2
13	32	$(8,16,16)$	$G(32,17)$	-	-	-	p-NA
13	32	$(8,16,16)$	$G(32,17)$	-	-	-	p-NA
13	36	$(4,18,36)$	$\mathbb{Z}_{36}$	-	-	-	C
13	36	$(6,12,12)$	$\mathbb{Z}_3 \times \mathbb{Z}_{12}$	-	-	-	A2
13	36	$(6,12,12)$	$G(36,6)$	-	-	-	S-NA-NP
13	36	$(6,12,12)$	$G(36,6)$	-	-	-	S-NA-NP
13	36	$(9,9,9)$	$G(36,3)$	-	-	-	S-NA-NP
13	39	$(3,39,39)$	$\mathbb{Z}_{39}$	-	-	-	C
13	40	$(4,10,20)$	$\mathbb{Z}_2 \times \mathbb{Z}_{20}$	-	-	-	A2
13	39	$(3,14,42)$	$\mathbb{Z}_{42}$	-	-	-	C
13	45	$(3,15,15)$	$\mathbb{Z}_3 \times \mathbb{Z}_{15}$	-	-	-	A2
13	48	$(3,12,12)$	$G(48,31)$	-	-	-	S-NA-NP
13	48	$(3,12,12)$	$G(48,33)$	-	-	-	S-NA-NP
13	48	$(4,6,12)$	$G(48,21)$	-	-	-	S-NA-NP
13	48	$(4,6,12)$	$G(48,31)$	-	-	-	S-NA-NP
13	48	$(6,6,6)$	$G(48,32)$	-	-	-	S-NA-NP
13	52	$(2,52,52)$	$\mathbb{Z}_{52}$	-	-	-	C
13	52	$(4,4,26)$	$G(52,1)$	-	-	-	S-NA-NP
13	54	$(2,27,54)$	$\mathbb{Z}_{54}$	-	-	-	C
13	56	$(2,28,28)$	$\mathbb{Z}_2 \times \mathbb{Z}_{28}$	-	-	Yes	A2
13	56	$(4,4,14)$	$G(56,6)$	-	-	-	S-NA-NP
13	60	$(5,5,5)$	$G(60,5)$	-	-	-	NS
13	64	$(2,16,16)$	$G(64,29)$	-	-	-	p-NA
13	64	$(2,16,16)$	$G(64,30)$	-	-	-	p-NA
13	64	$(2,16,16)$	$G(64,31)$	-	-	-	p-NA
13	64	$(4,4,8)$	$G(64,8)$	-	-	-	p-NA
13	64	$(4,4,8)$	$G(64,9)$	-	-	-	p-NA
13	64	$(4,4,8)$	$G(64,9)$	-	-	-	p-NA
13	64	$(4,4,8)$	$G(64,18)$	-	-	-	p-NA

Table 17: Separability of Surfaces of Genus 13 (Cont.)

σ	$\|G\|$	(l,m,n)	Group	p	q	r	Type
13	64	$(4,4,8)$	$G(64,20)$	-	-	-	p-NA
13	64	$(4,4,8)$	$G(64,21)$	-	-	-	p-NA
13	64	$(4,4,8)$	$G(64,32)$	-	-	-	p-NA
13	64	$(4,4,8)$	$G(64,33)$	-	-	-	p-NA
13	64	$(4,4,8)$	$G(64,33)$	-	-	-	p-NA
13	72	$(2,12,12)$	$G(72,21)$	-	-	-	S-NA-NP
13	72	$(2,12,12)$	$G(72,27)$	-	-	-	S-NA-NP
13	72	$(2,9,18)$	$G(72,16)$	-	-	-	S-NA-NP
13	72	$(3,4,12)$	$G(72,42)$	-	-	-	S-NA-NP
13	72	$(3,6,6)$	$G(72,44)$	-	-	-	S-NA-NP
13	72	$(3,6,6)$	$G(72,47)$	-	-	-	S-NA-NP
13	72	$(3,6,6)$	$G(72,47)$	-	-	-	S-NA-NP
13	72	$(4,4,6)$	$G(72,45)$	-	-	-	S-NA-NP
13	78	$(2,6,39)$	$G(78,4)$	-	-	-	S-NA-NP
13	90	$(2,6,15)$	$G(90,7)$	-	-	-	S-NA-NP
13	96	$(3,4,6)$	$G(96,3)$	-	-	-	S-NA-NP
13	96	$(3,4,6)$	$G(96,68)$	-	-	-	S-NA-NP
13	96	$(3,4,6)$	$G(96,70)$	-	-	-	S-NA-NP
13	104	$(2,4,52)$	$G(104,5)$	-	-	-	S-NA-NP
13	112	$(2,4,28)$	$G(112,13)$	Yes	Yes	-	S-NA-NP
13	120	$(2,5,10)$	$G(120,35)$	-	-	-	NS
13	128	$(2,4,16)$	$G(128,71)$	-	-	-	p-NA
13	128	$(2,4,16)$	$G(128,79)$	-	-	-	p-NA
13	144	$(2,4,12)$	$G(144,115)$	-	-	-	S-NA-NP
13	144	$(3,3,6)$	$G(144,184)$	-	-	-	S-NA-NP
13	180	$(3,3,5)$	$G(180,19)$	-	-	-	NS
13	288	$(2,3,12)$	$G(288,1024)$	-	-	-	S-NA-NP
13	360	$(2,3,10)$	$G(360,121)$	-	-	-	NS

7 Conjectures

Based on the data gathered from the computation of the separability of all tilings by triangles from genus 2 through 13 we believe that the following conjectures will hold true in general all though we currently lack a proof.

Conjecture 1. *For every genus $\sigma \geq 2$ there exists a tiling by $16(\sigma + 1)$, $(2, 4, 2\sigma + 2)$-triangles such that the tiling separates along the p and q reflections.*

Reason. The reasoning for this conjecture comes from a simple observation that there are two primary lines of symmetry on a torus, horizontal and vertical, and for every genus thus far there has existed a tiling group satisfying the conditions. □

To understand the second conjecture some preliminary work needs to be done in the theoretical implications of non-separability. To begin with we revisit the idea of closing a surface along the mirror of a reflection, as in Definition 15.

Definition 18. Let S be a surface with an embedded graph Γ such that S does not separate along some reflection R and let the associated mirror be M. Then define the surface Γ^* by taking one component of the surface $S - M$ and for each oval in M closing the surface with a open unit disk. Note that we will consider all the vertices along the oval, to belong to the new surface Γ^*. In addition, the seams between the component and the unit disks, will become edges connecting adjacent vertices in the mirror.

Lemma 23. *The number of vertices in Γ^* under a reflection R is $|V_\Gamma| + |V_M|$.*

Proof. Since the surface does not separate the vertices along the mirror will be counted twice and no vertices will be lost thus there are $|V_\Gamma| + |V_M|$ vertices in Γ^*. □

Lemma 24. *The number of edges in Γ^* under a reflection R is $|E_\Gamma| + 2\,|V_M|$.*

Proof. Since Γ does not separate along R, none of the edges will be lost and the edges along the mirror M will be counted twice, thus there are $|E_\Gamma| + 2\,|V_M|$ edges in Γ^*. □

Lemma 25. *The number of faces in Γ^* under a reflection R is $|F_\Gamma| + |V_M| + |\Theta|$.*

Proof. Since every face along the mirror M is split in two and there are no faces removed, there are at least $|F_\Gamma| + |V_M|$ face in Γ^*. Furthermore since a face is added for every oval there are $|F_\Gamma| + |V_M| + |\Theta|$ face in Γ^*. □

Definition 19. The genus of Γ^* is σ_{Γ^*}.

Theorem 6.

$$2(\sigma_S - \sigma_{\Gamma^*}) = |\Theta|\,.$$

Proof.

$$
\begin{aligned}
\chi(\Gamma^*) &= |V_{\Gamma^*}| + |F_{\Gamma^*}| - |E_{\Gamma^*}| \\
2 - 2\sigma_{\Gamma^*} &= |V_{\Gamma^*}| + |F_{\Gamma^*}| - |E_{\Gamma^*}| \\
2 - 2\sigma_{\Gamma^*} &= |V_\Gamma| + |V_M| + |F_\Gamma| + |V_M| + |\Theta| - |E_\Gamma| - 2\,|V_M| \\
2 - 2\sigma_{\Gamma^*} &= |V_\Gamma| + |F_\Gamma| - |E_\Gamma| + |\Theta| \\
2 - 2\sigma_{\Gamma^*} &= 2 - 2\sigma_S + |\Theta| \\
2(\sigma_S - \sigma_{\Gamma^*}) &= |\Theta|\,.
\end{aligned}
$$

□

Corollary 3 (Theorem 6). *If* $|\Theta| \equiv 1 \pmod 2$ *then* Γ^* *is non-orientable.*

Proof. Let $|\Theta| = 2n + 1$. Then by Theorem 6,

$$
\begin{aligned}
2(\sigma_S - \sigma_{\Gamma^*}) &= |\Theta| \\
2\sigma_S - 2\sigma_{\Gamma^*} &= 2n + 1 \\
2\sigma_S - 2n - 1 &= 2\sigma_{\Gamma^*} \\
\sigma_S - n - \frac{1}{2} &= \sigma_{\Gamma^*}.
\end{aligned}
$$

Thus σ_{Γ^*} is non-integer, and hence Γ^* is non-orientable. □

Conjecture 2. *If there exists a surface S with a tiling by triangles that separates along p, q, and r then S has genus 0.*

Reason. Suppose that you had a non-spherical surface that separated along two of the reflections. The most obvious lines of reflection on the surface for these to be are the horizontal and vertical lines of symmetry of the surface. Assume that the two separating reflection lie along those lines of symmetry, the mirror of the third reflection would wind it's way through the surfaces holes resulting in a non-orientable capping of the surface, and hence a non-separating reflection. □

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