3. Research Problems

3.1 Warm up problems

The following warm-up problems are intended to deepen the understanding of the development in Chapter 2. Some of the further reading from Sections 3.3, 3.4, and 3.5 is useful in solving these problems. Note that the problems may require you to look up material, make some additional assumptions, or make a cleaner statement of the problem.

The actual REU research problems are in the next section.

3.1.1 Basic problems

The problems concern some basic ideas about the group of isometries and its action on the geometric structure of S.

Problem W3.1. Why are the tilings called kaleidoscopic?

Problem W3.2. The isometries of a surface form a group.

Problem W3.3. The subgroup G is the subgroup of all orientation preserving isometries in G^* and the index of G in G^* is 2.

Problem W3.4. Vertices are G-equivalent if and only if they are G^*-equivalent. The same is true for edges.

Problem W3.5. The automorphism θ is unique.

Problem W3.6. Assume that G^* acts simply transitively on the triangles. Let P denote the vertex of the master tile opposite p, and Q and R similarly defined. Then the stabilizers of P are:

$$\begin{aligned} G_P^* &= \{g \in G^* : gP = P\} = \langle q, r\rangle = \langle r\rangle \ltimes \langle b\rangle \text{, and} \\ G_P &= \{g \in G : gP = P\} = \langle b\rangle . \end{aligned}$$

The first is dihedral and second is cyclic. Determine the other stabilizers of vertices.

Problem W3.7. For $g \in G^*$, $G^*_{gP} = gG^*_P g^{-1}$ and $G_{gP} = gG_P g^{-1}$.

Problem W3.8. Let e be a edge of the tiling, let r_e be the reflection in that edge, and let $g \in G^*$. Then $r_{ge} = g\, r_e g^{-1}$. Suggestion: Use the following fact about isometries. If g and h are isometries and $g = h$ on a small patch of the surface then $g = h$ everywhere on the surface (if S is connected!).

Problem W3.9. If two or more of l, m, and n are odd then all reflections in the edges of a tile are conjugate. Suggestion: Look at a vertex.

Problem W3.10. Find all the $(4,4,4)$-tilings on a surface of genus 3 with $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ as orientation-preserving automorphism group.

Problem W3.11. Up to isometry find all the triangular tilings on surfaces of genus 4 with Alt_5 as orientation-preserving automorphism group. Suggestion: use Magma.

3.1.2 Transitivity and conjugacy problems

In many of these problems we assume that G^* acts transitively on the triangles (in fact it is generally true that G^* acts simply transitively). Transitivity for connected surfaces follows from problem *W3.16.*

Problem W3.12. It is generally true that the number of tiles is $2|G|$. Prove this in the case of a scalene triangle. Suggestion: prove that G^* acts simply transitively on the triangles.

Problem W3.13. The tiling may have additional symmetries to those in G^* if the triangle is isosceles.

Problem W3.14. Assume that G^* acts simply transitively on the triangles. Each edge is equivalent to exactly one of the edges p, q, r of the master tile.

Problem W3.15. Assume that G^* acts simply transitively on the triangles. Then each vertex is equivalent to exactly one of the vertices P, Q, R of the master tile.

Problem W3.16. We may travel from one triangle to another by taking a *reflective walk* on the surface. Starting at the master tile, let Δ_0, $\Delta_1 = g_1\Delta_0$, $\ldots$, $\Delta_k = g_k\Delta_0$ be a sequence of tiles on S such that Δ_i and Δ_{i+1} have an edge in common for $i = 0, \ldots, k-1$. (By our notation we have assumed that we can get from the master tile to any other tile by transforming by a group element g_k. This problem will show how to inductively compute the g_k.) Pictorially, draw a curve from Δ_0 to Δ_k without passing through any vertices. Let $e_1, \ldots, e_k$, be the sequence of edges we encounter as we traverse the curve. Suppose that each e_i is equivalent to some edge f_i in the master tile - one of p, q, r, considered as edges. Let u_i be the reflection in f_i - again one of p, q, r, considered as reflections. Prove that $\Delta_k = u_1 \cdots u_k\Delta_0$ so that we may define g_k by $g_k = u_1 \cdots u_k$. Note that the inductive proof also shows to find the f_i.

Problem W3.17. What topological condition on the surface guarantees that G^* acts transitively on the tiles, using the above line of argument.

Problem W3.18. Problem *W3.14* has the following interpretation. According to the argument we may label every edge of the tiling with a p, q, or r, depending on which edge it is equivalent to. Take a reflective walk from the master tile to the desired tile. The desired group element is gotten by writing down the labels of the edges as we pass through them, in the order that we pass through them.

Problem W3.19. Prove all the facts on fixed points, mirrors, ovals in the first part of Section 3.4.

3.1.3 Construction and isomorphism problems

In classifying surface geometries and tilings by group actions many different generating triples give rise to essentially the same tiling. We want to be able to find out when two such tilings are isometrically equivalent. First we need to be able to construct the tiling.

Problem W3.20. Let G and θ be as given above and assume that equations 2.3 and 2.4 hold. Show that a surface can be constructed such that G and G^* are the corresponding groups of symmetries. The tiling can be constructed as follows. Take a disjoint set of triangles $\{\Delta_g : g \in G^*\}$ labelled by the elements of G^*. Label the edges of each triangle by p, q, and r and each of the vertices by P, Q, and R, ensuring that the labeling is consistent with the vertex angles. Now join the triangles Δ_g and Δ_h along edge p if $h = gp$, and similarly for the other edges. You need to be able to show that the object so constructed is a surface. If θ does not exist then a surface with G-symmetry may be constructed but this is more difficult.

Problem W3.21. Two generating triples (a, b, c), (a', b', c') are $\mathrm{Aut}(G)$-equivalent if there is an automorphism $\omega \in \mathrm{Aut}(G)$ such that $(a', b', c') = \omega \cdot (a, b, c) = (\omega(a), \omega(b), \omega(c))$. Pick the automorphism θ' associated to (a', b', c') to be $\omega \circ \theta \circ \omega^{-1}$. Show that θ' has the appropriate properties.

Problem W3.22. Prove Theorems 4.1 and 4.3. If the genus is not equal to 1 then you will need to use the AAA congruency theorem from non-euclidean geometry.

3.1.4 Universal cover problems

The universal cover is introduced in section 3.3.

Problem W3.23. This problem shows how to use the universal cover (Section 6) of the tiling and the group G to answer geometric questions about geodesics.

- In the $(2, 5, 5)$-tiling of the hyperbolic plane in Figure 3.2, pick a master tile and label a number of triangles by the group element (word in p, q, r) required to get from the master tile to the given tile. Suggestion: Use the idea of reflective walks in Problem *W3.10*, applied to the universal cover (Section 3.3).

- Pick an edge e of the master tile and determine the “smallest” transformation w (as a transformation of the universal cover) required to move the master tile to another tile such that the image edge lies on the same geodesic as e.

- Write w as a word in a, b, and c.

- Pick one of the surfaces S produced in problem *W3.12*. Compute the element $\bar{w} \in G$ corresponding to w using the specific generating triple. What is the action of $\bar{w}$ on the geodesic on S determined by e?

- *Punch line*: How long is the oval on S determined by e, in terms of the lengths of the sides of the basic triangle? What is the length of any oval in the tiling on S.

3.2 REU research problems

Each of the technical reports of the previous participants has a section on further research problems. These are an additional source of problems, though they have not been screened for “doability” (accessible to undergraduates).

3.2.1 Tiling enumeration

The tilings and tiling groups for small genus are known for genus $\sigma \leq 13$. The tiling groups were determined by [2] (conformal tiling groups in genus 2 and 3), REU students Ryan Vinroot [29] (most of genus 4 and 5) and Robert Dirks and Maria Sloughter [14] (most of genus 6 and 7), and finally for all $\sigma \leq 13$, by all these authors [9]. Additional applications of the low genus classification of tilings to the Teichmüller and moduli spaces of curves are described in Chapter 9.

Problem R3.1 Determine all the low genus tilings by quadrilaterals, pentagons, ... - say for $\sigma \leq 13$.

Problem R3.2 What proportion of surfaces admitting an (l, m, n)-group of conformal automorphisms G, also admit an (l, m, n)-tiling? The results in [9] indicate that the non-symmetric surfaces (no tiling) are relatively rare. A more basic question might be to find infinite families of tileable and non-tileable surfaces.

Problem R3 What is a fast algorithm to determine when an (l, m, n)-triple admits an involutary automorphism θ leading to a tiling? A direct algorithm that is fairly fast was developed by Dirks and Sloughter [14].

3.2.2 Oval intersections and (combinatorial) oval lengths

During the summer of 1996 by Schmidt [23], worked on oval lengths and oval intersection problems. In the summer of 2000 McCance and Weissman [22] continued some of the work of Schmidt in determining oval lengths for a large class of metacyclic groups. Section 3.4 develops some of the relations between the combinatorics of ovals and the group G. Note that in problems *R3.4* and *R3.6* do not require that the automorphism θ exists, i.e. that the tiling may be locally kaleidoscopic, with G as symmetry group (see Remark 2.5)

Problem R3.4. In the sphere, two ovals always meet in exactly 2 points, whereas for the torus we have examples of ovals meeting in exactly one point. What happens in the hyperbolic case? D. Schmidt produced a family of examples where the intersections were large. Part of this problem is to determine the number of edges in an oval.

Problem R3.5. Solve a special case of Problem R4 by solving the following extended problem. Let θ_1, θ_2 be two reflections in G^*. Let $\mathcal{O}_1^1, \ldots, \mathcal{O}_r^1$ be the ovals of θ_1 and $\mathcal{O}_1^2, \ldots, \mathcal{O}_s^2$ be the ovals of θ_2. Suppose for the moment that θ_1, θ_2 commute. This means that if $\mathcal{O}_i^1$, intersects $\mathcal{O}_j^2$, then they must meet at right angles and must meet in 1 or 2 points. Here's the question. What is the incidence graph of the two sets of ovals?

Problem R3.6. Let $\mathcal{O}_1, \ldots, \mathcal{O}_r$ be the set of ovals passing through a vertex. What is the pattern of oval intersections $\#\mathcal{O}_i \cap \mathcal{O}_j$. For example, what group characteristics e.g., simple group, divisibility conditions on the group order, the stabilizer of P has prime order, will guarantee that the number of intersection points is constant?

3.2.3 Oval lengths, geodesics, the length spectrum, and billiards

See the technical reports of Derby-Talbot [13], Woods [28], and Lehman-White [20].

Problem R3.7. Find classes of surfaces with more than one length of ovals on S. The tricky case is equilateral triangles with all of l, m, n even.

Problem R3.8. Are the ovals ever the shortest (closed) geodesics on the surface? For Klein's quartic curve (genus 3 with a (2,3,7)-tiling or Hurwitz tiling) this has been answered in the negative [13], and also in the negative for the three genus 14 Hurwitz tilings [28]. This problem motivated the study of the length spectrum of a surface described next.

Problem R3.9 The length spectrum of a surface is the sequence of all lengths of closed geodesics on a surface, in increasing order. Determine the lower end of the length spectrum of a surface with a kaleidoscopic tiling. Fairly extensive results were found for the genus 3 Hurwitz surface [13], and the genus 14 surfaces by [28]. The main problems would be to find efficient ways to generate the length spectrum and then to generate and compare initial portions of the spectrum for a family of groups, e.g., Hurwitz groups.

Problem R3.10 A tool to analyzing the length spectrum is the following. There is a exact sequence $\Gamma \to \Lambda \to G$, where Γ encodes the geodesics on S and Λ lifts the G-action to the universal cover. The geodesics can be determined by computing traces of elements of Γ. An entire family of (l, m, n) groups can be studied by looking at the traces of hyperbolic elements in Λ. Pick a triple (l, m, n) and analyze the "universal length spectrum" coming from Λ. These ideas are discussed in [13] and [28].

Problem R3.11 The isometric images of a geodesic has the same length as the original. Therefore, there will be several geodesics of the same length on a surface if there are any symmetries. Are there geodesics of the same length which are not images of each other by a surface isometry? More generally, describe the multiplicity function on the length spectrum of a surface. The first problem also makes sense for the universal length spectrum.

Problem R3.12 A geodesic on a tiled surface is cut up into a sequence of segments by the lines of the tiling. Each of these segments can be reflected back to the master tile to form a "billiard path". If we shoot a billiard in a hyperbolic billiard table in the shape of a tile on a hyperbolic surface, the ball will travel around the polygon obeying the law of reflection each time it hits a wall. It is not difficult to see that if we do not constrain the ball to reflect, it will describe a geodesic on the surface. The geodesic will have finite length if and only if the corresponding billiard path returns to its initial point and direction. The picture of the billiard path is independent of the surface and really only depends on the hyperbolic triangle. Thus the closed geodesics can be viewed as closed billiard paths. Describe, draw and analyze the small closed billiard paths. The lengths of all billiard paths is simply the universal length spectrum descried above. (See also the next problem). Lehman and White [20] have worked on this problem for Hurwitz tilings.

Problem R3.13 If a closed billiard path on hyperbolic triangle is straightened out on a surface, it does not generally form an entire geodesic. The billiard will determine an element $\gamma \in \Lambda$. Specifically, the billiard, when straightened out on the universal cover, will describe a straight line segment from x_0 to γx_0, where x_0 is a point on the billiard path. If $g \in G$ is the image of γ in G then $o(g)$ repetitions of the billiard path will close up on the surface. Thus the length of the of the geodesic is $o(g)$ times the length of one cycle of the billiard path. Several questions now arise.

- For a given billiard path, what is the smallest $o(g)$ possible?
- What are the possible $o(g)$'s?
- Is is possible to have two different surfaces S_1 and S_2 and two different billiard paths B_1 and B_2 such that for surface S_1 the geodesic corresponding to billiard path B_1 is longer than the geodesic corresponding to B_2, but on S_2 the opposite is true?

3.2.4 Divisible Tilings

The basic problems here are more of a combinatorial nature though lots of good group theory questions are lurking in the background. The tiling on the torus derived from the tiling by squares on the plane has the following property (see Section 3.5). Each square can be subdivided into triangles (put in the diagonals) so that the new tiling is a geodesic, kaleidoscopic tiling. You can also achieve the same effect in the universal covering of this tiling, as pictured in Figure 3.1. Now extend this situation to the hyperbolic case, i.e., find all quadrilateral tilings in the hyperbolic plane, such that they may be subdivided into a finer tiling by triangles. This reduces to finding all quadrilaterals that can be tiled by triangles. For example, in Figure 3.2 one can easily see that 12 adjacent $(2,5,5)$-triangles can be put together to form a $(5,5,5,5)$-quadrilateral. This problem has been worked on by Haney and McKeough [18], and Smith [25]. Their combined work along with the contribution of Broughton is in [8] completely solve the problem of divisible tilings in the plane. The problem of which surfaces have divisible tilings has not been solved.

Problem R3.14. Suppose that we have a quadrilateral Q that can be tiled by a triangle T. Under what conditions does a tiling of a surface by triangles T also admit a compatible tiling by quadrilaterals isometric to Q. Also answer the question with the roles of T and Q reversed. First step: find interesting examples. There should always be an infinite family of examples.

Problem R3.15. Find a surface of minimum genus surface supporting both tilings. Is there anything interesting about the symmetry group. Haney and McKeough found some results in [18]

3.2.5 Separating reflections

Problem R3.16. A reflection is called *separating* if by cutting along all the ovals of the mirror we separate the surface into two pieces, otherwise it is called non-separating. Find an *efficient* group-theoretic or combinatorial algorithm which determines if a reflection is separating. Previous REU work has been done by Jim Belk [11], and Baeth, Deblois, and Powell [1].

Problem R3.17. In [4] it is shown how to convert the separability problem into a problem in solving linear equations in a group algebra. The methods work in a straight forward way for abelian G. Extend the methods to non-abelian G, say split metacyclic groups.

3.2.6 Harder Problems

Just in case we run out of easy ones!

Problem R3.18. The tiling by $(2,4,4)$-triangles may be realized as follows. Tile the plane by $(2,4,4)$-triangles and then cut out a large square in the plane compatible with the tiling as in Figure 3.1). Identify the opposite sides of the of the square to form a torus. The square in Figure 3.1 is called a fundamental region and $G = \mathbb{Z}_2 \ltimes \mathbb{Z}_{16} \times \mathbb{Z}_{16}$. Now extend this idea to hyperbolic tilings. The question comes in two parts:. First find criteria for deciding what types fundamental regions are permissible, second find a group theoretic algorithm for computing the region. There are infinitely many selections for a fundamental region. In fact any collection of G^*-inequivalent tiles will do. Here are some possibilities:

- the fundamental region should be a convex polygon,
- the image of the boundary in S should have some nice properties,
- the fundamental region should be computable by a simple algorithm with G^*.

Problem R3.19 If $(l, m, n) = (2, 3, 7)$ then G is called a Hurwitz group since the order of the group reaches its maximal size given in the Hurwitz equation. The corresponding surfaces are called Hurwitz surfaces. In [12] the symmetries of all Hurwitz surfaces of the form $PSL_2(q)$, $q = p$ or p^3, where p is a prime were determined. It was proven that all symmetries are conjugate, and that with few exceptions in the list of the first few hundred primes the number of ovals in the mirror of a symmetry is 5 or fewer and most commonly it is one. There are a few exceptions, the worst known case being $p = 967$ where the number of ovals is 44. Here's the problem: Explain the discrepancies in the list.

3.2.7 Some additional problems

Some additional problems requiring more background are given at the end of Chapters 7, 8 and 9.

3.3 Geometry in the universal covering space

We did not describe where the geometry came from in euclidean and the hyperbolic cases. Tilings of the torus can be first constructed in the plane and then mapped to the torus. Consider the tiling of the plane obtained by first constructing a tiling by squares by drawing in horizontal and vertical lines that meet the coordinate axes at the integer points and then drawing in both diagonals for each square (see Figure

3.1 below for a portion of this tiling). Next, consider the map $\mathbb{R}^2 \to \mathbb{R}^4 = \mathbb{C}^2$ given by $(x, y) \to (e^{2\pi ix/m}, e^{2\pi iy/m})$. The image of the plane is a torus $S' \subset \mathbb{R}^4$. The edges of the tiling in the plane map onto the tiling on the torus and the reflections in the plane induce reflections on the image torus. For instance the reflection $x \to k - x$ gets mapped to the transformation $(w, z) \to (e^{2\pi ik/m}\bar{w}, z)$. Clearly this map is an isometry of S'. The projection $\mathbb{R}^4 \to \mathbb{R}^3$ which carries S' to its image S in the figures is a map of the type $(u, v, x, y) \to ((R + rx)u, (R + rx)v, ry)$, which unfortunately doesn't preserve in $\mathbb{R}^3$ the geometric structure that we had in $\mathbb{R}^4$. The composite map $q : \mathbb{R}^2 \to S$ is called the universal covering map and it allows us to create the geometry of the tiling on the surface from the geometry of the tiling on the universal cover $\mathbb{R}^2$, rather than from the embedding $S \subset \mathbb{R}^3$.

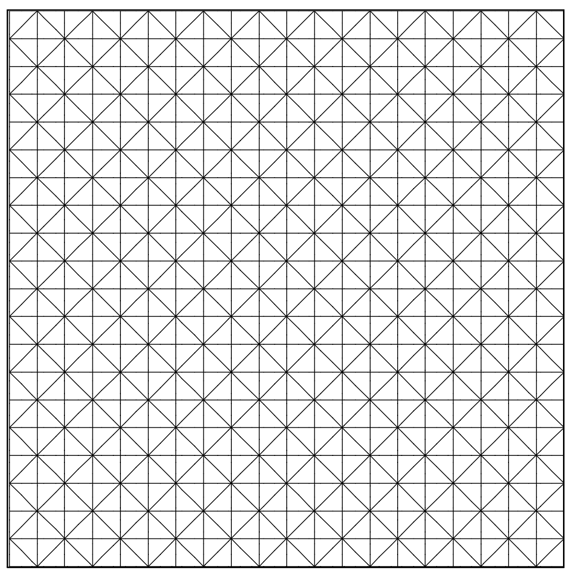

Figure 3.1. 2-4-4 tiling of the plane

For surfaces of genus $\sigma \geq 2$ there is also a universal cover $\mathbb{H}$ with a tiling and the corresponding reflections, though the realization of the map $q : \mathbb{H} \to S$, is much more difficult. The image surface lies in higher dimensional space. The universal cover is the so-called Poincaré or unit disk in $\mathbb{C}$ with the following geometry.

Geometry of the hyperbolic plane

lines	circles perpendicular to the unit circle or diameters
distance	a complicated formula involving log and crossratios
angles	the standard angle between curves
reflections	$z \to \frac{c\bar{z}-1}{\bar{z}-\bar{c}}$, where c is the center of the reflecting circle

As in the Euclidean case tilings defined on $\mathbb{H}$ project to tilings defined on S. We do not need to know what the map is, but just that it exists, since we can do all our geometric constructions in $\mathbb{H}$. A picture of a portion of the tiling by $(2, 5, 5)$-triangles on $\mathbb{H}$ is given in Figure 3.2, and one by $(2, 3, 7)$-triangles is given in Figure 3.3.

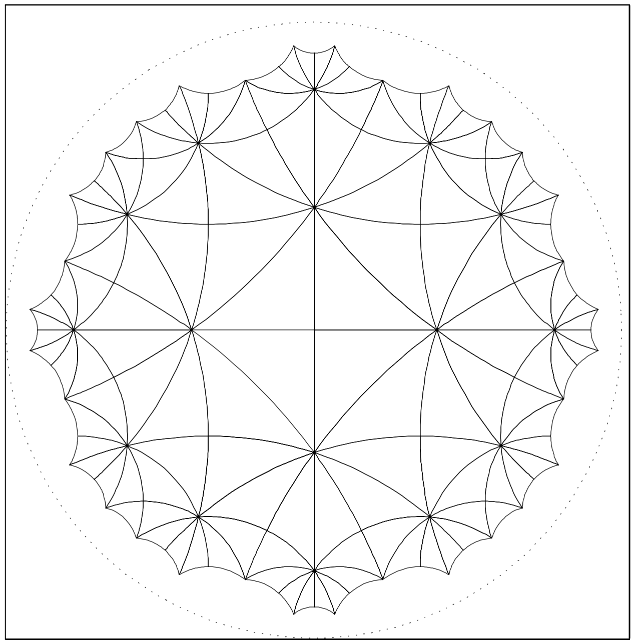

Figure 3.2. 2-5-5 tiling of hyperbolic plane

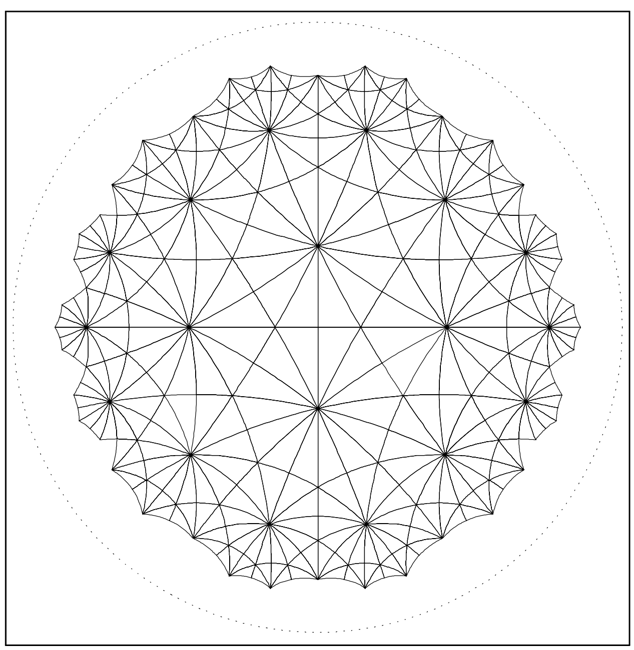

Figure 3.3. 2-3-7 tiling of hyperbolic plane

The sphere is its own universal cover and has the following geometric constructs.

Geometry of the sphere

lines	great circles
distance	great circle distance
angles	the standard angle between curves
reflections	reflections in planes passing through the center

The covering $q : U \to S$ of a surface by its universal cover preserves geodesics, distances, angle and area, at least in the small. That means that for each point x on the surface there is a neighbourhood V of x such that each component U of $q^{-1}(V)$ is mapped isometrically by q to its image $q : U \leftrightarrow V$. Thus $q^{-1}(V)$ is an infinite disjoint union of isometric copies of the same set U (for $\sigma \geq 1$). This property can be extended from neighbourhoods to a polygon in an arbitrary tiling as long as the polygon does not meet itself along edges or vertices. In this case $q^{-1}(V)$ will be an infinite set of disjoint congruent polygons (try to visualize this in the figures). It follows that each tiling of a surface can be lifted to the universal covering space (unwrapping) and each tiling of the universal covering space be wrapped into possibly infinitely many different tiled surfaces. In the hyperbolic case, infinitely many genera are produced. The ideas of wrapping and unwrapping can be most easily understood by considering the torus examples. The hyperbolic wrappings are not so easily visualized.

By means of the universal cover we have introduced three simply connected geometries, the sphere, the euclidean plane and the hyperbolic plane. Much of the standard geometry of the euclidean plane holds true, especially congruence theorems. What does not hold true are the parallel postulate (no parallels in the sphere and infinitely many in the hyperbolic plane). As a consequence, in the sphere and hyperbolic planes the angle sum of a triangle is not π but as given in the table on angle sums. A consequence of this is the AAA congruence theorem in spherical and hyperbolic geometry: if the corresponding angles of two triangles are congruent, then the triangles are congruent. In the euclidean case we only get similarity.

3.4 Fixed points, mirrors, ovals and stabilizers

Here we collect some facts on stabilizers of ovals and vertices of the tiling. These should be helpful in answering the problems on oval intersections. We assume all the previous notation, namely a surface tiled by (l, m, n)-triangles, and $G, G^*, a, b, c, p, q, r,$ and θ have all of their previous interpretations. For $x \in S$ let $G_x = \{g \in G : gx = x\}$, and for a reflection $\phi \in G^*$, let $S_\phi = \{x \in S : \phi x = x\}$ denote the mirror of ϕ. Here are some facts about and fixed points, mirrors, ovals and their stabilizers.

Fixed points:

- $G_{gx} = gG_xg^{-1}$, for $g \in G^*$, $x \in S$.

- Let P, Q, R be the vertices of the master tile, as given in Problem *W6*. Then,

$$G_P = \langle b \rangle \,,\ G_Q = \langle c \rangle \,,\ G_R = \langle a \rangle \,.$$

- The element $g \in G$ fixes a vertex $x \in S$ if and only if one the following holds:

$$\begin{aligned} x &= uP, g \in u\langle b\rangle u^{-1}, \text{for some } u \in G, \\ x &= vQ, g \in v\langle c\rangle v^{-1}, \text{for some } u \in G, \\ x &= wR, g \in w\langle a\rangle w^{-1}, \text{for some } w \in G. \end{aligned}$$

Mirrors, ovals and reflections

- Each reflection $\phi \in G^*$ is conjugate to one of p, q, r.
- If any two of p, q, r meet at an angle π/k, where k is an odd integer, then the two reflections are conjugate.
- p and q are conjugate if and only if $a^{-1} = \theta(g)g^{-1}$ for some g, r and q are conjugate if and only if $b = \theta(h)h^{-1}$ for some h, and p and r are conjugate if and only if $b = \theta(k)a^{-1}k^{-1}$ for some k.
- For each $g \in G^*$, $gS_\phi = S_{g\phi g^{-1}}$,
- Each mirror is a disjoint union of ovals: $S_\phi = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_k$.
- If $g \in G^*$ centralizes ϕ, then g permutes the ovals of ϕ.
- If $g \in G^*$ maps an oval of ϕ to another oval of ϕ, then $g \in \mathrm{Cent}_{G^*}(\phi)$.
- $\mathrm{Cent}_{G^*}(\phi) = \langle\phi\rangle \times \mathrm{Cent}_G(\phi)$.

The dissected boundary If we take the boundary of the master tile and cut it at all the vertices of even order, then we either get a circle or one, two or three segments. We call this the dissected boundary of the master tile, $\partial^d(\Delta_0)$. From the facts above it follows that the reflections in any two edges belonging to the same component of the dissected boundary conjugate. We may order the vertices so that we have the following 4 cases:

Case	Algebraic Description	Geometric Description
OOO	l, m, n odd	circle
EOO	l even, m, n odd	one segment $RPQR$
EEO	l, m even, n odd	two segments, RP, PQR
EEE	l, m, n even	three segments,RP, PQ, PR

The remaining four cases OEO, OOE, OEE, EOE are similar to the above.

Let $\mathcal{O}$ be a oval of the reflection ϕ. Then the edges comprising $\mathcal{O}$ must come from exactly one of the components B of the dissected boundary $\partial^d(\Delta_0)$, and ϕ is conjugate to each of the reflections in $\{p, q, r\}$ determined by B. Each oval of ϕ must correspond to exactly one of the components of $\partial^d(\Delta_0)$, though usually there is at most one such component. Here are some additional facts on ovals and stabilizers. Let B denote one of the components of the dissected boundary $\partial^d(\Delta_0)$ and let $\mathcal{O}(\phi, B)$ denote the set of ovals of ϕ that correspond to B. We also have the following:

- $\mathrm{Cent}_{G^*}(\phi)$ permutes $\mathcal{O}(\phi, B)$ transitively.
- The number of ovals is given by:

$$|\mathcal{O}(\theta, B)| = \frac{|\mathrm{Cent}_G(\phi)|}{|\mathrm{Stab}_G(\mathcal{O})|}.$$

- If B is a circle then the stabilizer of the oval is cyclic

$$\mathrm{Stab}_G(\mathcal{O}) \simeq \mathbb{Z}_M$$

and if B is a segment then stabilizer of the oval is dihedral.

$$\mathrm{Stab}_G(\mathcal{O}) \simeq D_M$$

The stabilizers in the four cases are given in the proposition below. First a bit of notation. Let $\mathcal{O}_p, \mathcal{O}_q, \mathcal{O}_r$ denote the ovals which bound the master tile in sides p, q, r respectively. Let h_p, h_q, and h_r denote a generator of the rotational subgroup of $\mathrm{Stab}_G(\mathcal{O}_p)$, $\mathrm{Stab}_G(\mathcal{O}_q)$, and $\mathrm{Stab}_G(\mathcal{O}_r)$, respectively. Let λ, μ, ν be integers such that

$$l = 2\lambda,\ m = 2\mu,\ n = 2\nu,$$

in the even cases and

$$l = 2\lambda + 1,\ m = 2\mu + 1,\ n = 2\nu + 1,$$

in the odd cases.

Then we have the following table:

Case	OOO	EOO	EEO	EEE
$\mathrm{Stab}_G(\mathcal{O}_p)$	$\langle a^{\lambda+1}b^{\mu+1}c^{\nu+1}\rangle$	$\langle a^{\lambda}, b^{\mu+1}c^{\nu+1}a^{\lambda}c^{\nu}b^{\mu}\rangle$	$\langle a^{\lambda}, b^{\mu}\rangle$	$\langle a^{\lambda}, b^{\mu}\rangle$
$\mathrm{Stab}_G(\mathcal{O}_q)$	$\langle b^{\mu+1}c^{\nu+1}a^{\lambda+1}\rangle$	$\langle b^{\mu}a^{\lambda}b^{\mu+1}, c^{\nu+1}a^{\lambda}c^{\nu}\rangle$	$\langle b^{\mu}, c^{\nu+1}a^{\lambda}c^{\nu}\rangle$	$\langle b^{\mu}, c^{\nu}\rangle$
$\mathrm{Stab}_G(\mathcal{O}_r)$	$\langle c^{\nu+1}a^{\lambda+1}b^{\mu+1}\rangle$	$\langle c^{\nu}b^{\mu}a^{\lambda}b^{\mu+1}c^{\nu+1}, a^{\lambda}\rangle$	$\langle c^{\nu}b^{\mu}c^{\nu+1}, a^{\lambda}\rangle$	$\langle c^{\nu}, a^{\lambda}\rangle$
h_p	$a^{\lambda+1}b^{\mu+1}c^{\nu+1}$	$a^{\lambda}b^{\mu+1}c^{\nu+1}a^{\lambda}c^{\nu}b^{\mu}$	$a^{\lambda}b^{\mu}$	$a^{\lambda}b^{\mu}$
h_q	$b^{\mu+1}c^{\nu+1}a^{\lambda+1}$	$b^{\mu}a^{\lambda}b^{\mu+1}c^{\nu+1}a^{\lambda}c^{\nu}$	$b^{\mu}c^{\nu+1}a^{\lambda}c^{\nu}$	$b^{\mu}c^{\nu}$
h_r	$c^{\nu+1}a^{\lambda+1}b^{\mu+1}$	$c^{\nu}b^{\mu}a^{\lambda}b^{\mu+1}c^{\nu+1}a^{\lambda}$	$c^{\nu}b^{\mu}c^{\nu+1}a^{\lambda}$	$c^{\nu}a^{\lambda}$

Analogous formulas hold for the four other cases OEO, OOE, OEE, EOE.

3.5 Divisible tilings

We call a tiling *divisible* if the tiling can be subdivided into a finer tiling. It suffices to tile the larger tile by the smaller tile, see the examples below. The hyperbolic case is the most interesting. If a (s, t, u)-triangle or an (s, t, u, v)-quadrilateral can be tiled by K $(l, m, n)-$triangles then each of l, m, n is a multiple of one of s, t, u, or v, and one of

$$(1 - \frac{1}{s} - \frac{1}{t} - \frac{1}{u}) = K(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n})$$

or

$$(2 - \frac{1}{s} - \frac{1}{t} - \frac{1}{u} - \frac{1}{v}) = K(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n})$$

holds because of angle and area considerations. Some examples are given in Figures 3.4, 3.5 and 3.6. below. Note that the $(5,5,5,5)$ quadrilateral can be subdivided twice, once into twelve $(2,5,5)$-triangles and then each $(2,5,5)$-triangle can be subdivided into two $(2,4,5)$-triangles - see Figures 3.5 and 3.6.

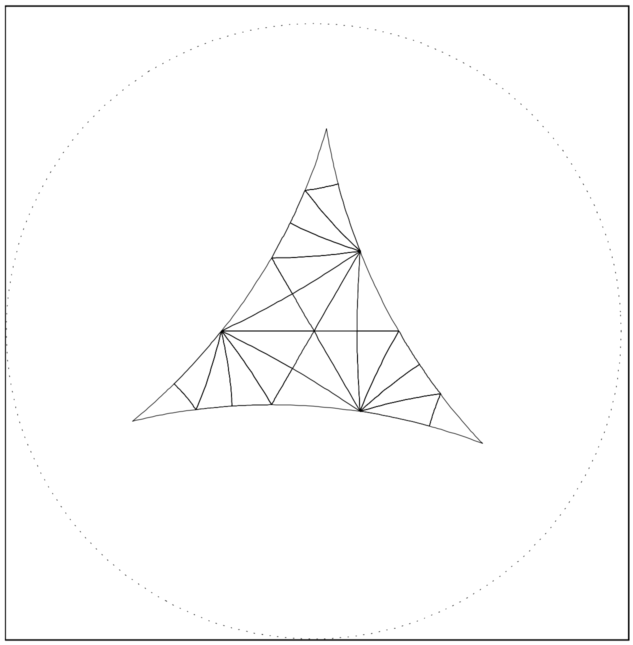

Figure 3.4 - $(2,3,7)$ tiling of $(7,7,7)$

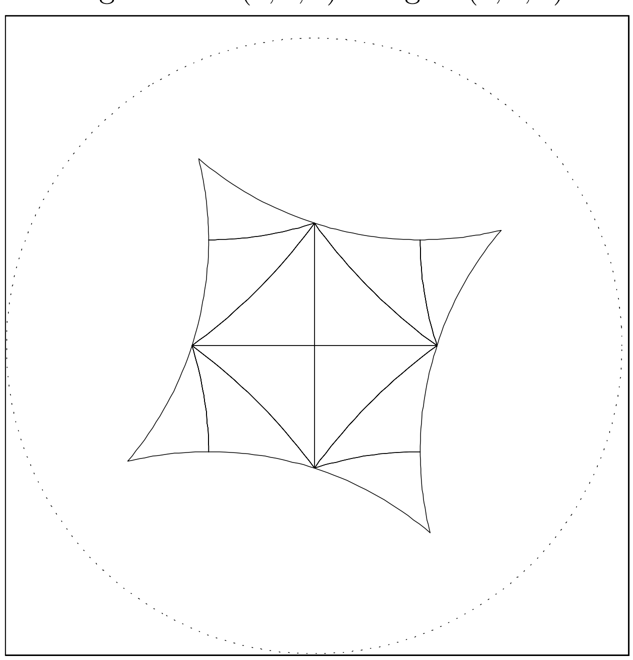

Figure 3.5 - $(2,5,5)$ tiling of $(5,5,5,5)$

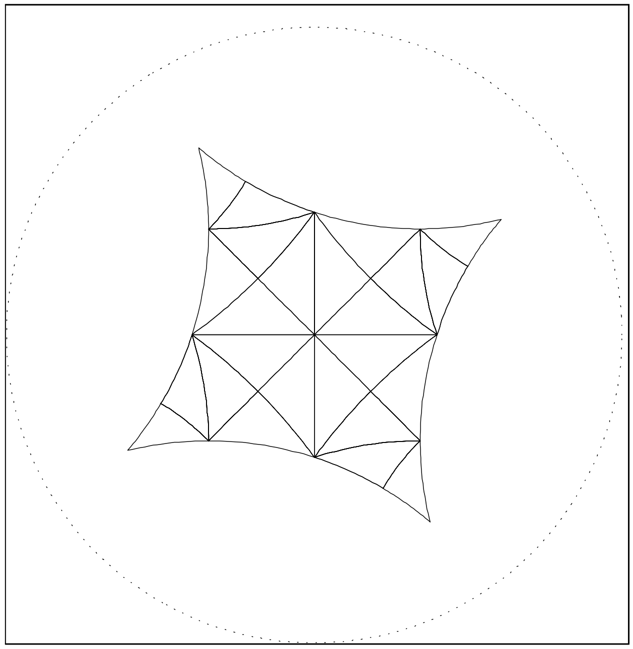

Figure 3.6 - $(2, 4, 5)$ tiling of $(5, 5, 5, 5)$