6. The Topology of Group Actions, Fundamental Groups, and Branched Covers

In this Chapter we will discuss quotient spaces obtained from group actions and their relation to fundamental groups. The fundamental group will be presented in three versions, the Galois definition, (Section 6.3), the standard homotopy definition (Section 6.4), and the path definition (Section 6.6), derived from tilings. The covering space notions will be generalized to branched covering of surfaces in Section 6.5, which will allow us to discuss non-Galois construction of tilings. Finally, non-kaleidoscopic tilings will be discussed in the context of unbranched coverings. See [24] and [17] for background references on the topology of fundamental groups and covering spaces.

6.1 Introductory example

The group $\Lambda = \mathbb{Z}^2$ acts on the Euclidean plane by translation: $(m, n) \cdot (x, y) = (x + m, y + n)$. It is easy to see that every point (x, y) in the plane is Λ-equivalent to a point (x', y') in the square $R = \{(u, v) : 0 \leq u \leq 1, 0 \leq u \leq 1\}$. We call R a fundamental region for the given action. Now, note that the points $(0, y)$ and $(1, y)$ are Λ-equivalent as well as the pairs $(x, 0)$ and $(x, 1)$. These points correspond to the left, right, bottom and top edges of the fundamental region respectively. If we identify the left and right edges and the top and bottom edges of the square we get a torus as the identification space. Furthermore, every orbit Λx corresponds to a unique point in the torus. In the fundamental region we have uniqueness only in the interior with possible non-uniqueness along the boundary.

6.2 Discontinuous groups, quotients and fundamental regions

Let X be a space with a decent topology and let Λ be a group of continuous transformations of X. Often X will be a metric space and Λ will be a group of isometries.

Definition 6.1 The action of Λ is *discontinuous* on X if for any two points $x, y \in X$ there are neighbourhoods U of x and V of y such that U meets at most a finite number of sets in the collection $\{gV : g \in \Lambda\}$. The group Λ acts *without fixed points* or *acts freely* or has a *free action* if the stabilizer of the point x, $\Lambda_x = \{g \in \Lambda : gx = x\}$, is trivial for all x. If Λ acts freely and discontinuously we say that the action is properly discontinuous.

Example 6.1 In our example we make take U to be any disc and $\{gV : g \in \Lambda\}$ to be a set of discs of the same small radius all centred at the points $\{(x_0 + m, y_0 + n) : m, n \in \mathbb{Z}\}$ for some (x_0, y_0). In fact, the action is properly discontinuous.

Remark 6.1 Note that the stabilizers of points Λ_x are automatically finite for a discontinuous action, such as in the case of triangle groups.

Definition 6.2 Let Λ act on the space X. The *quotient space* or *orbit space* is the set of orbits $\{\Lambda x : x \in X\}$. The canonical projection is the map $q : X \to X/\Lambda$ given by $x \to \Lambda x = \overline{x}$. The topology on X/Λ is the quotient topology in which we declare $U \subseteq X/\Lambda$ to be open if and only if $q^{-1}(U)$ is open in X.

Remark 6.2 Suppose X is a metric space so that we may define the topology by convergence of sequences. Then $\overline{x_n} \to \overline{x_0}$ if and only if there are $g_n \in \Lambda$ such that $g_n x_n \to x_0$.

Definition 6.3 A *fundamental region* for the Λ-action on X is a closed set R such that:

-   the boundary ∂X of X is nowhere dense,

-   each orbit Λx meets R in at least one point,

-   if Λx meets the interior R° then it meets R in exactly one point in the interior.

In the case above and in the cases we will consider R will be a closed region in the plane, sphere or disc, bounded by a finite number of smooth curves. Each orbit meets R either in a unique point in the interior or perhaps several points on the curves forming the boundary. Moreover, for each boundary curve we will usually be able to find an element of Λ that maps the curve to another curve on the boundary. Thus by "sewing up the boundaries" we will be able to form the quotient space. This was illustrated in our example above.

Example 6.2 Let Λ^* be the group generated by the reflections in a Schwartz triangle in one of the geometries. Then any tile is a fundamental region for the Λ^*-action. In this case each orbit intersects the tile in exactly one point and hence the tile (with its three edges and vertices) is a model of the quotient space as well. Now let Λ be the subgroup of index 2 of orientation preserving elements of Λ^*, and consider two adjacent tiles as in Figure 5.1. They form a fundamental region for Λ, (since Λ has index 2 in Λ^*) but now some boundary identifications are required. The curve labeled p can be rotated onto its mirror image by $a^{-1} \in \Lambda$ so they need to be sewed together. Similarly $b \in \Lambda$ rotates r onto its mirror image. After all the identifications are complete the resulting quotient space is homeomorphic to a sphere, the three edges are joined together to form an equator, and the three vertices are uniformly spaced on this equator. In transforming the fundamental region to the quotient space, the angle $\frac{2\pi}{l}$ at the point labelled a is stretched by a factor of l to form a full circular neigbourhood of the image point in the quotient space. This corresponds to the fact that the order of the stabilizer subgroup $\langle a \rangle$ is l. Similar comments apply to the two other branch points.

Proposition 6.1 *Suppose that Λ acts properly discontinuously on the region X in the plane. Then, the quotient space X/Λ is a surface. If Λ consists entirely of orientation-preserving maps then X/Λ is orientable.*

Remark 6.3 We have not imposed enough conditions to imply that X/Λ is compact or that it has only a finite number of handles.
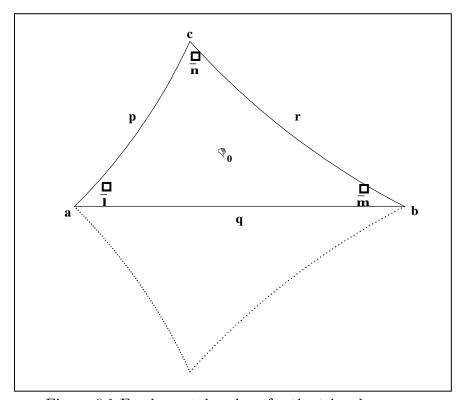

Figure 6.1 Fundamental regions for the triangle groups

6.3 Covering spaces and Galois groups

Covering spaces First we shall define and give some examples of covering spaces. It is the basic underlying construction for almost all of our work.

Definition 6.4 A map $p : X \to Y$ is called a *covering space* if for each point $y_0 \in Y$ there is an open neighbourhood V of y_0 which is *evenly covered* by p, i.e., $p^{-1}(V)$ is a disjoint union of open sets each of which is mapped homeomorphically onto V. In particular, p is a local homeomorphism. If $y \in Y$ then $p^{-1}(y)$ is called the *fibre above y*. If Y is connected then the fibres all have the same cardinality. This common cardinality is called the *degree of the covering*.

Example 6.3 The map of the plane onto the torus $p : \mathbb{R}^2 \to T$ given by $p(x, y) = (e^{2\pi i x}, e^{2\pi i y})$ is a covering map. Note that this is essentially the same map as the covering space achieved by a group action in our introductory example.

Example 6.4 Let the torus be modeled by $T = \{(z, w) \in \mathbb{C} : |z| = |w| = 1\}$. Then the map $(z, w) \to (z^m, w^n)$ is a mn to 1 covering space.

Example 6.5   Let $p : \mathbb{C} \to \mathbb{C}$ be the map $p(z) = z^3 - 3z$. It is easy to prove that p is an even covering of a neighbourhood of $w \in \mathbb{C}$ if $p(z) - w$ has three distinct roots. If $p(z) - w$ has a multiple root then $p(z) - w$ and $p'(z)$ have a common root. The roots of $p'(z) = 3(z^2 - 1)$ are ± 1. Therefore the bad w's are $-2 = p(1)$ and $2 = p(-1)$. The bad fibres are $p^{-1}(-1) = \{1, -2\}$, (solve $z^3 - 3z + 2 = 0$), and $p^{-1}(1) = \{-1, 2\}$, (solve $z^3 - 3z - 2 = 0$). Thus the restricted map $p : \mathbb{C} - \{\pm 1, \pm 2\} \to \mathbb{C} - \{\pm 1\}$ is a branched cover. Obviously this analysis can be performed for any polynomial or rational function. This will be discussed in greater detail in Section 6.5.

Proposition 6.2   *Suppose that Λ acts properly discontinuously on a sufficiently nice space X (e.g., a region in the plane). Then, the canonical projection $q : X \to X/\Lambda$ is a covering space.*

Definition 6.5   A space is *simply-connected* if every loop in the space can be shrunk to a point. A universal covering space X of a space Y is a simply-connected space X and a covering space projection $p : X \to Y$. Normally we just refer to the space X as the universal covering space.

Remark 6.4   The map $p : \mathbb{R}^2 \to T$ defined above is a universal covering space.

Proposition 6.3   *If Y is a connected, sufficiently nice space then there is an essentially unique universal covering space $p : X \to Y$. I.e., given two universal covering spaces $p_1 : X_1 \to Y$, and $p_2 : X_2 \to Y$, then there is homeomorphism $h : X_1 \to X_2$ such that $p_1 = p_2 \circ h$. In diagram form:*

$$
\begin{array}{ccc}
X_1 & \xrightarrow{h} & X_2 \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\
Y & \xrightarrow{id} & Y
\end{array}
$$

Galois Groups   The notion of a Galois group is the basic link between group actions and the fundamental group. In Section 6.5 the relation to Galois groups of field extensions will be explained.

Definition 6.6   If $p : X \to Y$ is a covering space then the *group of covering transformations* or *Galois group* $G = \mathrm{Gal}(X/Y)$ of p is the set of all homeomorphisms h of X satisfying: $p \circ h = p$. In diagram form:

$$
\begin{array}{ccc}
X & \xrightarrow{h} & X \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
Y & \xrightarrow{id} & Y
\end{array}
$$

Example 6.6   The group of translations $\mathbb{Z}^2$ acting on $\mathbb{R}^2$ as described in the first

section is the group of covering transformations since

$$
\begin{aligned}
p((m,n) \cdot (x,y)) &= p(x+m, y+n) \\
&= (e^{2\pi i(x+m)}, e^{2\pi i(y+n)}) \\
&= (e^{2\pi ix}, e^{2\pi iy}) = p(x,y).
\end{aligned}
$$

Likewise the maps $(z,w) \to (\alpha z, \beta w)$ where $\alpha^m = \beta^n = 1$ are covering transformations of the torus covering given in Example 6.4.

Fix $x_0 \in X$ and $y_0 \in Y$ such that $p(x_0) = y_0$. Note that the equation $p \circ h = p$ implies that $h(x_0)$ is another point of $p^{-1}(y_0)$. It turns out that in the connected case that h is uniquely determined by the value of $h(x_0)$. For example the group element of $\mathbb{Z}^2$ is uniquely determined by the image $(0,0) \in p^{-1}(1,1)$. Thus G determines a permutation group of the fibre $p^{-1}(y_0)$ and has no fixed points on $p^{-1}(y_0)$ except for the identity transformation. It follows that G acts freely on X.

Definition 6.7 A connected covering space $p : X \to Y$ is said to be a *regular* or *Galois covering* if the group of covering transformations is transitive on fibres of p.

Example 6.7 The covering spaces of the torus described above are all regular or Galois coverings.

Proposition 6.4 *If $p : X \to Y$ is a regular covering space then the group of covering transformations G acts properly discontinuously on X and the regular covering space p is equivalent to the quotient mapping $q : X \to X/G$. More specifically there is a homeomorphism $h : Y \leftrightarrow X/\Gamma$ such that the diagram*

$$
\begin{array}{ccc}
X & \xrightarrow{\ id\ } & X \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
Y & \xrightarrow{\ h\ } & X/G
\end{array}
$$

commutes.

Proposition 6.5 *The universal covering space $p : X \to Y$ is regular covering.*

Definition 6.8 We shall call the Galois group $\mathrm{Gal}(X/Y)$ of the universal covering space the *fundamental group* of Y and denote it by $\Gamma = \Gamma_Y$. We give two alternative constructions of the fundamental group in sections 6.4 and 6.6.

Proposition 6.6 *Let $r : Y \to Z$ be a connected covering space Then Y and Z have the same universal covering space with the commutative diagram.*

$$
\begin{array}{ccc}
X & \xrightarrow{\ id\ } & X \\
\downarrow{\scriptstyle p_Y} & & \downarrow{\scriptstyle p_Z} \\
Y & \xrightarrow{\ r\ } & Z
\end{array}
$$

Furthermore, the fundamental groups Γ_Y and Γ_Z satisfy $\Gamma_Y \subseteq \Gamma_Z$, and the diagram above is equivalent to:

$$
\begin{array}{ccc}
X & \xrightarrow{\ id\ } & X \\
\downarrow{\scriptstyle q_Y} & & \downarrow{\scriptstyle q_Z} \\
X/\Gamma_Y & \xrightarrow{\ q_{Y/Z}\ } & X/\Gamma_Z
\end{array}
$$

The relative projection $q_{Y/Z}$ is defined by the inclusion of orbits $\Gamma_Y x \to \Gamma_Z x$ and the fibres of the map $q_{Y/Z}$ may be identified with the left coset space Γ_Z/Γ_Y. The degree of the cover is $|\Gamma_Z : \Gamma_Y|$. Finally r is a Galois cover if and only if $\Gamma_Y \trianglelefteq \Gamma_Z$ and in this case $\mathrm{Gal}(Y/Z) \simeq \Gamma_Z/\Gamma_Y$.

Remark 6.5 The last proposition can be illustrated for tori. Let $X = \mathbb{R}^2$ let p_Y and p_Z be the maps:

$$
\begin{aligned}
p_Y(x, y) &= (\exp(\frac{2\pi i x}{m}), \exp(\frac{2\pi i y}{n})), \\
p_Z(x, y) &= (\exp(2\pi i x), \exp(2\pi i y)), \\
r(z, w) &= (z^m, w^n).
\end{aligned}
$$

The groups are given by

$$
\begin{aligned}
\Gamma_Y &= m\mathbb{Z} \times n\mathbb{Z} \subseteq \mathbb{Z} \times \mathbb{Z} = \Gamma_Z, \\
\Gamma_Z/\Gamma_Y &= \mu_m \times \mu_m
\end{aligned}
$$

where μ_m the group of m'th roots of unity.

Proposition 6.7 *Let $r : Y \to Z$, be a covering of finite degree $d = |\Gamma_Z : \Gamma_Y|$, where Y and Z are closed surfaces of genus σ and σ' respectively. Then,*

$$
2\sigma - 2 = d(2\sigma' - 2).
$$

In particular if $\Gamma_Y \trianglelefteq \Gamma_Z$ so that $H = \Gamma_Z/\Gamma_Y$ is the Galois group of p and $Z = Y/H$ then

$$
2\sigma - 2 = |H|\,(2\sigma' - 2).
$$

A proof for surfaces, valid for the more general case of branched coverings is given in Section 6.5.

6.4 Topological fundamental group, topology of coverings

For the development below, a space must be "sufficiently nice", see [27]. All our spaces under consideration will easily satisfy this hypothesis. Let Y be a connected, "sufficiently nice" space. If y_0 is a distinguished point in Y then $\pi_0(Y, y_0)$ is the set of homotopy classes of closed paths based at y_0. The multiplication of two paths is given by their concatenation; if $\alpha, \beta \in \pi_0(Y, y_0)$ then $\alpha * \beta$ is α followed by β. The inverse of a path is gotten by traversing the path in reverse direction. Here are some basic facts on the fundamental group.

Proposition 6.8   *The fundamental group is a group. If Y is path connected then $\pi_0(Y, y_0)$ and $\pi_0(Y, y_1)$ are isomorphic, via the homomorphism induced by the map $\alpha \to \delta^{-1} * \alpha * \delta$, where δ is any path from y_0 to y_1. If $f : (X, x_0) \to (Y, y_0)$ is a map of pointed spaces so that $f(x_0) = y_0$, then the map $f_* : \pi_0(X, x_0) \to \pi_0(Y, y_0)$, induced by $(f_* \alpha)(t) = f(\alpha(t))$, $0 \leq t \leq 1$, is a homomorphism. For a composition of maps $(p \circ q) : (X, x_0) \to (Y, y_0) \to (Z, z_0)$, we have the functorial property $(p \circ q)_* = p_* q_*$.*

Lifting criterion and monodromy

Definition 6.9   Let $p : (X, x_0) \to (Y, y_0)$ be a connected covering space, suppose that Z is path connected, and that $f : (Z, z_0) \to (Y, y_0)$ is any map. Then, we say that f lifts to map $\widetilde{f} : (Z, z_0) \to (Y, y_0)$ provided that $p \circ \widetilde{f} = f$. In diagram form we have:

$$
\begin{array}{ccc}
 & & (X, x_0) \\
 & {}^{\widetilde{f}} \nearrow & \downarrow{}^{p} \\
(Z, z_0) & \xrightarrow{\ f\ } & (Y, y_0)
\end{array}
\tag{1}
$$

The key example is $Z = [0, 1]$ and $z_0 = 0$, so that f is a path in Y starting at y_0. It is easy to show that the lift of a path always exists and is unique. Furthermore, it can also be shown that a homotopy of paths α_1 and α_2, fixing the start point, lifts to a homotopy of the lifts $\widetilde{\alpha_1}$ and $\widetilde{\alpha_2}$.

If $\alpha \in \pi_0(Y, y_0)$ represents a homotopy class of loops then we can consider whether or not the lift $\widetilde{\alpha}$ closes up, i.e., satisfies $\widetilde{\alpha}(0) = \widetilde{\alpha}(1) = x_0$. By lifting homotopies, it is not hard to show that if one path in a homotopy class has a closed lift then they all have closed lifts. Thus, we may identify $p_*(\pi_0(X, x_0))$ with the subgroup of loops in $\pi_0(Y, y_0)$, whose lifts starting at x_0 are closed. Moreover, if the images $p_*(\widetilde{\alpha_1})$ and $p_*(\widetilde{\alpha_2})$ are homotopic then their lifts are homotopic. This implies that p_* is injective. We record all these observations as a proposition.

Proposition 6.9   *Let $p : (X, x_0) \to (Y, y_0)$ be a connected covering space. Then, $p_* : \pi_0(X, x_0) \to \pi_0(Y, y_0)$ is injective and the image $p_*(\pi_0(X, x_0))$ is the subgroup of all loops whose lifts to X, starting at x_0 are closed. The map $\alpha \to \widetilde{\alpha}(1)$ give a 1-1 correspondence between the cosets $p_*(\pi_0(X, x_0)) * \alpha$ and the fibre of $p^{-1}(y_0)$.*

If Z is an arbitrary path-connected space we can extend the above proposition to a general lifting criterion.

Proposition 6.10   *Suppose that Z is a path-connected space and that we have maps as in the following diagram.*

$$
\begin{array}{ccc}
 & & (X, x_0) \\
 & & \downarrow{}^{p} \\
(Z, z_0) & \xrightarrow{\ f\ } & (Y, y_0)
\end{array}
\tag{2}
$$

Then the map f can be lifted as in 6.1 if and only if $f_*(\pi_0(Z, z_0)) \subseteq p_*(\pi_0(X, x_0))$. If the map lifts then it is unique.

Proof. We sketch the proof since it will help our later development. The necessity of the condition follows from the fact that $f_* = p_* \widetilde{f}_*$ as group homomorphisms. To construct $\widetilde{f}$, pick a path α from z_0 to z, let $\widetilde{f_* \alpha}$ be the lift of the image path $f_* \alpha$ in Y, and define $\widetilde{f}(z) = \widetilde{f_* \alpha}(1)$. If α_1 is another path from z_0 to z then $f_*(\alpha^{-1} * \alpha_1)$ lies in $f_*(\pi_0(Z, z_0)) \subseteq p_*(\pi_0(X, x_0))$ and hence the lift $\widetilde{f_*(\alpha^{-1} * \alpha_1)}$ is a closed path in $\pi_0(X, x_0)$. This implies that $\widetilde{f_* \alpha}(1) = \widetilde{f_* \alpha_1}(1)$, and thus $\widetilde{f}(z)$ is well defined. The continuity of $\widetilde{f}$ is easily shown using small paths, and the uniqueness follows from the uniqueness of path lifting. ∎

Remark 6.6 One format of the lifting criterion is to lift maps of spaces to their covers. In diagram form we have:

$$
\begin{array}{ccc}
(W, w_0) & \dashrightarrow & (X, x_0) \\
\downarrow q & & \downarrow p \\
(Z, z_0) & \xrightarrow{\ h\ } & (Y, y_0)
\end{array}
.
$$

We wish to fill in the dotted arrow with a lift $\widetilde{h}$. In this case we take $f = h \circ q$ and the lifting criterion becomes $h_* q_*(\pi_0(W, w_0)) \subseteq p_*(\pi_0(X, x_0))$.

The lifting criterion allows us to determine the existence of covering transformations.

Corollary 6.11 *Consider the diagram to determine the existence of a covering transformation h of the covering space $p : X \to Y$ satisfying $h(x_0) = h(x_1)$*

$$
\begin{array}{ccc}
(X, x_0) & \dashrightarrow & (X, x_1) \\
\downarrow p & & \downarrow p \\
(Y, y_0) & \xrightarrow{\ id\ } & (Y, y_0)
\end{array}
.
$$

The covering transformation h exists if and only if $p_*(\pi_0(X, x_0)) = p_*(\pi_0(X, x_1))$. Furthermore, if h exists, it is a homeomorphism and is unique, and if $x_0 = x_1$ then h is the identity map.

Path lifting allows us to define the monodromy representation of a cover $p : X \to Y$. Let $\alpha \in \pi_0(Y, y_0)$, let $x \in p^{-1}(y_0)$ be arbitrary, and $\widetilde{\alpha}_x$ the lift of α starting at x. By varying x we get a permutation $\mu_\alpha : p^{-1}(y_0) \to p^{-1}(y_0)$ defined by $\mu_\alpha(x) = \widetilde{\alpha}_x(1)$. We also denote $\mu_\alpha(x)$ by x^α, it is only defined if $p(x) = \alpha(0)$. Let us comment on different ways of writing the monodromy action and then state it main properties the next proposition. We shall need the following notation. Let J be a set. Then $\Sigma(J)$ is the group of permutations of the set J with composition of mappings as the group multiplication.

Remark 6.7 The symmetric group on n symbols is often realized so that function composition goes from right to left, performing the rightmost operation first. For example if a is the permutation given by $1 \to 2, 2 \to 1, 3 \to 3$ and b by $1 \to 1, 2 \to 3, 3 \to 2$ then $b \circ a$ is given by $1 \to 2, 2 \to 1, 3 \to 3$. This is compatible with

the left operator action notation $(b \circ a) \cdot x = b \cdot (a \cdot x) = b(a(x))$. Alternatively, the symmetric group elements are sometimes written in cycle notation with multiplication left to right so that $ab = (1,2)(2,3) = (1,2,3)$. This is compatible with the right exponentiation action notation $x^{ab} = x \cdot (ab) = (x \cdot a) \cdot b = (x^a)^b$. In the second format the map $\pi_0(Y, y_0) \to \Sigma(p^{-1}(y_0))$ is a homomorphism though the action should be written x^α. In particular the Magma uses the later realization of the symmetric group. Further note, that for any group G there is an opposite group G^{op} with the same underlying set and opposite multiplication operator $*$, defined by $a * b = ba$. This the relation between the two interpretations of $\Sigma(p^{-1}(y_0))$.

Proposition 6.12 *Let $p : X \to Y$ be a connected covering space, $y_0 \in Y$, and $\mu : \pi_0(Y, y_0) \to \Sigma(p^{-1}(y_0))$ the monodromy representation defined by $\alpha \to \mu_\alpha$ defined above. Then, we have the following.*

1. The map μ is an anti-homomorphism, when $\Sigma(p^{-1}(y_0))$ is considered with the composition multiplication, i.e., $\mu_{\alpha*\beta} = \mu_\beta \circ \mu_\alpha$.

2. The map μ is an homomorphism, when $\Sigma(p^{-1}(y_0))$ is considered with left to right multiplication compatible with a right action, i.e., $x^{\alpha*\beta} = (x^\alpha)^\beta$.

3. The image of $\pi_0(Y, y_0)$ in $\Sigma(p^{-1}(y_0))$ is a transitive subgroup of $\Sigma(p^{-1}(y_0))$.

4. If the diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\widetilde{h}} & X_2 \\
\downarrow{p_1} & & \downarrow{p_2} \\
Y_1 & \xrightarrow{h} & Y_2
\end{array}
$$

   is a transformation of covering spaces then

$$\widetilde{h}(x^\alpha) = (\widetilde{h}(x))^{h_*\alpha}$$

   when $p(x) = \alpha(0)$. If $\widetilde{h} = g \in \mathrm{Gal}(X/Y)$ then, this becomes

$$g(x^\alpha) = (gx)^\alpha. \tag{3}$$

Proof. The anti-homomorphism property is easily verify by lifting the product loop $\alpha * \beta$. To prove transitivity let x_1 and x_2 lie in $p^{-1}(y_0)$ and let β be a path from x_1 to x_2 that exists because X is connected. Then β is the lift of $\alpha = p_*\beta$ starting at x_1 and hence $x_1^\alpha = x_2$. The rest of the proof is fairly straightforward. ∎

Remark 6.8 Let δ a path from y_1 to y_2 in Y. The $\alpha \to \delta^{-1} * \alpha * \delta$ defines an isomorphism $\delta_* : \pi_0(Y, y_1) \to \pi_0(Y, y_2)$. Furthermore path lifting along δ defines a map $\kappa : p^{-1}(y_1) \to p^{-1}(y_2)$. The map κ intertwines the monodromies of $\pi_0(Y, y_2)$ and $\pi_0(Y, y_1)$, namely $\mu_{\delta_*(\alpha)} = \kappa^{-1} \circ \mu_\alpha \circ \kappa$.

Proposition 6.13 *Let $p : X \to Y$ be a connected covering space, let $y_0 \in Y$, and let $x \in p^{-1}(y_0)$. Then,*

$$p_*(\pi_0(X, x)) = \{\alpha \in \pi_0(Y, y_0) : x^\alpha = x\},$$

and if $x_1^\alpha = x_2$ then

$$p_*(\pi_0(X, x_2)) = \alpha^{-1} p_*(\pi_0(X, x_1))\alpha. \tag{4}$$

Hence

$$\ker \mu \;=\; \bigcap_{x \in p^{-1}(y_0)} \{\alpha \in \pi_0(Y, y_0) : x^\alpha = x\}$$

$$=\; \bigcap_{\alpha \in \pi_0(Y, y_0)} \alpha^{-1} p_*(\pi_0(X, x_0))\alpha.$$

Galois correspondence We are now ready to describe the Galois correspondence between connected covering spaces of (Y, y_0) and subgroups of $\pi_0(Y, y_0)$. Let $\mathcal{C}$ denote the set of (connected) covering spaces $p : (X, x_0) \to (Y, y_0)$ where two covering spaces $p_1 : (X_1, x_1) \to (Y, y_0)$ and $p_2 : (X_2, x_2) \to (Y, y_0)$ are identified if there is an homeomorphic equivalence mapping e such that

$$
\begin{array}{ccc}
(X_1, x_1) & \xrightarrow{\;e\;} & (X_2, x_2) \\
\downarrow{p_1} & & \downarrow{p_2} \\
(Y, y_0) & \xrightarrow{\;id\;} & (Y, y_0)
\end{array}
$$

Let $\mathcal{P}$ denote the set of all subgroups of $\pi_0(Y, y_0)$. There is a 1-1 correspondence between $\mathcal{C}$ and $\mathcal{P}$ such that $p : (X, x_0) \to (Y, y_0)$ corresponds to $\Pi \subseteq \pi_0(Y, y_0)$ if and only if

$$\Pi = p_*(\pi_0(X, x_0)). \tag{5}$$

First we note that if two covering spaces determine the same subgroup then, by Remark 6.6, there are maps $e_1 : (X_1, x_1) \to (X_2, x_2)$ and $e_2 : (X_2, x_2) \to (X_1, x_1)$, that satisfy $p_1 = p_2 e_1$ and $p_2 = p_1 e_2$. It then follows that $e_1 e_2$ satisfies $e_1 e_2(x_2) = x_2$ and $p_2 e_1 e_2 = p_1 e_2 = p_2$. By Remark 6.6 $e_1 e_2$ is the identity transformation of X_1. Similarly $e_2 e_1$ is the identity transformation of X_2 and hence e_1 and e_2 are homeomorphisms. Secondly, note that all the conjugates of the subgroup Π in equation (6.5) are obtained by picking different base points in the fibre $p^{-1}(y_0)$ of a single covering space $p : X \to Y$, according to Proposition 6.13.

Our next order of business to construct a cover from a subgroup. We do that in the next proposition.

Proposition 6.14 *Let Y be a connected space such that each point in Y has a simply-connected open neighbourhood. Let $y_0 \in Y$ and $\Pi \subseteq \pi_0(Y, y_0)$ be any subgroup. Then there is a covering space $p : (X, x_0) \to (Y, y_0)$ satisfying 6.5. The covering space is unique up to equivalence.*

A sketch of the general proof is given here, a more specific construction is given at the end of section 6.5, for branched covers of the sphere.

Proof. A full technical proof may be found in [27], we just give the main idea, which is to unwind all paths in Y except those lying in Π. To this end, let $W = P(Y, y_0)$ be the space of paths in Y based at y_0, and w_0 the trivial loop $w_0(t) = y_0$, $0 \le t \le 1$. Now define the endpoint map from $e : W \to Y$ by $e(w) = w(1)$. If w_1 and w_2 are two points of w that have the same endpoints, i.e., $e(w_1) = e(w_2)$ then $w_1 * w_2^{-1} \in \pi_0(Y, y_0)$. Define an equivalence relation on W by $w_1 \sim w_2$ if and only if $e(w_1) = e(w_2)$ and $w_1 * w_2^{-1} \in \Pi$. Let X be the quotient space $W/\sim$ with quotient map $w \to \overline{w}$, $x_0 = \overline{w_0}$ and $p : X \to Y$ be the projection $\overline{w} \to e(w)$. It may be verified that $p : (X, x_0) \to (Y, y_0)$ is a covering space. Now suppose that $\alpha(t) \in \pi_0(Y, y_0)$. The lift to X starting at x_0 is the path $\widetilde{\alpha}(t) = \overline{w^t}$, $0 \le t \le 1$, where $w^t(s) = \alpha(st), 0 \le s, t \le 1$. For $p(\overline{w^t}) = e(w^t) = \alpha(1 \cdot t) = \alpha(t)$, and $\overline{w^0} = \overline{w_0} = x_0$ by construction. By definition $\widetilde{\alpha}$ is closed if and only if $\widetilde{\alpha}(1) = x_0$, which in turn happens if and only if $e(w^1) = e(w^0)$ and $\alpha = w^1 * (w^0)^{-1} \in \Pi$. Since $p_*(\pi_0(X, x_0))$ is set of all loops in $\pi_0(Y, y_0)$ that lift to a closed loop, then 6.5 is satisfied. ∎

Corollary 6.15 *Let $\mu : \pi_0(Y, y_0) \to \Sigma(J)$ be a (right) transitive permutation representation on an index set J. Then there is a covering space $p : X \to Y$ and an identification of the fibre $p^{-1}(y_0)$ with J so that the monodromy representation is the original μ.*

Proof. Pick $j_0 \in J$ and let $\Pi = \{\alpha \in \pi_0(Y, y_0) : j_0^\alpha = j_0\}$. The cosets $\Pi\alpha$ are in 1-1 correspondence to J. Let $p : (X, x_0) \to (Y, y_0)$ be the covering space corresponding to Π. The points of the fibre $p^{-1}(y_0)$ are also in 1-1 correspondence with the cosets $\Pi\alpha$. The original permutation representation and the monodromy representation are both equivalent to the right action on the cosets $\Pi\alpha$. Thus we may take the identification $j \to \Pi\alpha \leftarrow x$ for $j \in J$ and $x \in p^{-1}(y_0)$ effect the equivalence of the two representations. ∎

The next three proposition give a formal statement of the Galois correspondence.

Proposition 6.16 *Let $\mathcal{C}$ and $\mathcal{P}$ be defined as above. Then, there is a 1-1 correspondence between covering spaces $p : X \to Y$ and subgroups $\Pi \subseteq \pi_0(Y, y_0)$ defined by 6.5. Furthermore,*

- *if $p_1 : (X_1, x_1) \to (Y, y_0)$, and $p_2 : (X_2, x_2) \to (Y, y_0)$ are in $\mathcal{C}$ then $p_*(\pi_0(X_1, x_1)) \subseteq p_*(\pi_0(X_2, x_2))$ if and only if there is a covering space $q : (X_1, x_1) \to (X_2, x_2)$ such that $p_1 = q \circ p_2$,*

- *$\Pi \trianglelefteq \pi_0(Y, y_0)$ if and only if $\Pi = p_*(\pi_0(X, x))$ for every $x \in p^{-1}(y_0)$, and*

- *$\Pi \trianglelefteq \pi_0(Y, y_0)$ if for any $\alpha \in \pi_0(Y, y_0)$ if one lift of α is closed then all the lifts of α starting at the various points of $p^{-1}(y_0)$ are closed.*

Proposition 6.17 *Let $p : (X, x_0) \to (Y, y_0)$ be a covering space, let $G = \mathrm{Gal}(X/Y)$ be the Galois group of p, and let N be the normalizer of $p_*(\pi_0(X, x_0))$ in $\pi_0(Y, y_0)$. Then there is a epimorphism $\eta : N \to G$ defined by*

$$x_0^\alpha = \widetilde{a_{x_0}}(1) = \eta(\alpha)x_0, \tag{6}$$

and

$$p_*(\pi_0(X, x_0)) \hookrightarrow N \twoheadrightarrow G$$

is exact. If $p : (X, x_0) \to (Y, y_0)$ *is regular then we get*

$$p_*(\pi_0(X, x_0)) \hookrightarrow \pi_0(Y, y_0) \twoheadrightarrow G$$

Proof. If $\alpha \in N$ and $x_1 = x_0^\alpha$, then $p_*(\pi_0(X, x_0)) = p_*(\pi_0(X, x_1))$ by 6.4. It follows that there a covering transformation $h : (X, x_0) \to (X, x_1)$ by 6.11. Let $\eta(\alpha) = h$. If $\alpha, \beta \in N$ then

$$\begin{aligned}
\eta(\alpha * \beta) &= x_0^{\alpha * \beta} \\
&= (x_0^\alpha)^\beta \\
&= (\eta(\alpha)x_0)^\beta \\
&= \eta(\alpha)(x_0^\beta) \\
&= \eta(\alpha)\eta(\beta)x_0.
\end{aligned}$$

The third line comes from 6.3. The remainder of the proof is fairly straight forward. ∎

Proposition 6.18 *Let notation be as in Proposition 6.17 and suppose that* $x_1 = gx_0$ *for* $g \in \mathrm{Gal}(X/Y)$. *Let* $\eta_0 : N \to G$ *and* $\eta_0 : N \to G$ *defined by*

$$\begin{aligned}
x_0^\alpha &= \widetilde{a_{x_0}}(1) = \eta_0(\alpha)x_0, \\
x_1^\alpha &= \widetilde{a_{x_1}}(1) = \eta_1(\alpha)x_1.
\end{aligned}$$

Then

$$\eta_1(\alpha) = g\eta_1(\alpha)g^{-1}, \alpha \in N. \tag{7}$$

Proof. The path $g\widetilde{a_{x_0}}$ is the lift $\widetilde{a_{x1}}$. Thus

$$\eta_1(\alpha)gx_0 = \eta_1(\alpha)x_1 = g\widetilde{a_{x_0}} = g\eta_0(\alpha)x_0$$

and so $\eta_0(\alpha)x_0 = g^{-1}\eta_1(\alpha)gx_0$, and 6.7 follows. ∎

Finally we need to construct Galois extensions of finite covers.

Proposition 6.19 *Let* $p : X \to Y$ *be a finite branched cover of degree* d, *let* $\mu : \pi_0(Y, y_0) \to \Sigma_d$ *be its monodromy representation. Then there is a finite Galois cover of* Y, $W \xrightarrow{q} X \xrightarrow{p} Y$, *factoring through* X, *such that the following holds.*

- $G = \mathrm{Gal}(W/Y)$ is isomorphic to image of the monodromy subgroup $\langle \mu_\alpha : \alpha \in \pi_0(Y, y_0) \rangle$ of Σ_d.

- $H = \mathrm{Gal}(W/X)$ is isomorphic to the subgroup $\langle \mu_\alpha : \alpha \in \pi_0(Y, y_0), x_0^\alpha = x_0 \rangle$ of Σ_d.

- The degree of $r = p \circ q$ is at most $d!$.

- There is a 1-1 order preserving correspondence between intermediate covers $(W, w_0) \xrightarrow{q} (Z, z_0) \xrightarrow{p} (Y, y_0)$ and subgroups of $H \subseteq G$, by $H \leftrightarrow \text{Gal}(W/Z)$.

- $(Z, z_0) \xrightarrow{p} (Y, y_0)$ Galois if and only if $\text{Gal}(W/Z) \trianglelefteq G$ and then

$$\text{Gal}(Z/Y) \simeq \text{Gal}(W/Y)/\text{Gal}(W/Z).$$

6.5 Branched coverings of surfaces.

Definition 6.10   Let $p : S \to T$ be a map of closed, connected surfaces. The map p is said to be holomorphic if for every point $z_0 \in S$ there are neighbourhoods V of z_0 and W of $w_0 = p(z_0)$, and homeomorphisms $\phi : V \to D$, $\psi : W \to D$, to the open unit disc D such that:

1. The map $\psi \circ p \circ \phi^{-1}$ is the function $z \to z^e$ where e is a nonnegative integer. The integer $e = e(p, z_0)$ is called the branching order or ramification degree of p at z_0. If $e > 1$ we say that p is ramified at z_0, that p ramified over $w_0 = p(z_0)$, and that w_0 is a branch point of p.

2. For any two such ϕ_1, ϕ_2 and ψ_1, ψ_2, the $\phi_1 \circ \phi_2^{-1}$ and $\psi_1 \circ \psi_2^{-1}$ are conformal maps of the complex plane to itself, on its domain of definition.

If p is a non-constant map then it is automatically surjective and the ramification degree is always positive. Such maps are called *branched coverings*.

Remark 6.9   Note that the family of maps $\{\phi_i\}$ and $\{\psi_i\}$ define a conformal structure on S and T (measurement of angles) and that p is conformal except when the branching order $e > 1$. The map p is ramified at z_0 if and only if the derivative $dp(z_0) = 0$. In fact the degree of ramification is the smallest integer e such that $\frac{d^n}{dz^n} f(z) = 0$, for $n = 1, \ldots, e - 1$, when p is written in any local coordinates $f(z) = \psi \circ p \circ \phi^{-1}(z)$.

Definition 6.11   For a branched cover $p : S \to T$ we define $\text{Gal}(S/T)$ to be the set of conformal automorphisms of S commuting with the projection, i.e., $\text{Gal}(S/T) = \{g \in \text{Aut}(S) : pg = p\}$

Example 6.8   Let $p : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the map $p(z) = z^3 - 3z$, discussed in Example 6.5. Since $p'(z) = 3(z^2 - 1) = 0$ at $z = \pm 1$, and $p''(z) = 6z \neq 0$ at $z = \pm 1$, then p is doubly ramifies at ± 1 in the finite plane. At $z = \infty$, let $\phi(z) = \psi(z) = 1/z$, Then

$$\psi \circ p \circ \phi^{-1}(z) = \frac{1}{z^{-3} - 3z^{-1}}$$
$$= \frac{z^3}{1 - 3z^2}.$$

and so p has ramification degree 3.

Here are some basic facts about branched coverings

Proposition 6.20 *Let $p : S \to T$ be a branched cover. Then, the following are true.*

1. For every $w \in T$

$$\sum_{z \in p^{-1}(w)} e(p, z) = d.$$

   where d is the degree of the cover.

2. The ramification degree, $e(p, z) > 1$ for at most finitely many points on S, and hence the set of branch points $B \subset T$ is also finite.

3. Let $T^\circ = T - B$, and $S^\circ = p^{-1}(T^\circ)$. Then, the restricted map $p : S^\circ \to T^\circ$ is a connected covering space of degree d.

4. If $g \in \text{Gal}(S^\circ/T^\circ)$ then g may be extended to a conformal automorphism in $\text{Gal}(S/T)$. Thus $\text{Gal}(S^\circ/T^\circ)$ and $\text{Gal}(S/T)$ are canonically isomorphic.

We will need a notation to describe the ramification information of a fibre. Suppose that the fibre $p^{-1}(w)$ has n_i points of ramification degree e_i for $i = 1, \ldots, k$ then we say that the fibre type of $p^{-1}(w)$ is $e_1^{n_1} \cdots e_k^{n_k}$. For example in Example 6.8 the unramified fires have type 1^3, there are two fibres of type $1 \cdot 2$ and one fibre of type 3. Note that $\sum_{i=1}^{k} n_i e_i = d$.

If S and T have tilings the we say that the tilings are compatible with $p : S \to T$ if each closed tile and closed edge of the tilings on S is mapped homeomorphically onto an image tile in T. It turns out that for a compatible tiling, ramification points must be vertices. Here is a way to construct compatible tiling pairs. Let $\mathcal{T}$ be a tiling on T be a tiling such that the branch points of T are vertices of the tiling. The pullback $p^{-1}(\mathcal{T})$ of the tiling is the tiling defined as follows. Let Δ be a tile on T. The inverse image $p^{-1}(\cdot^\circ)$ is a disjoint union of open sets on S each of which is homeomorphic to $\cdot^\circ$. The tiles are lifted homeomorphically since $\cdot^\circ$ is simply connected. Likewise the interiors of edges are lifted homeomorphically. The vertices in $p^{-1}(\mathcal{T})$ are simply the inverse images of the vertices on T. In now follows that each closed tile in $p^{-1}(\mathcal{T})$ maps homeomorphically to a tile in T, except possibly at vertices. A simple topological argument, using the continuity of p, and the fact that the polygons in T do not self-intersect at the vertices show that the closure of a lifted tile does not self-intersect in S. Thus the pull back consists of tiles each of which is homeomorphically mapped onto a tile in T. Note that since p is conformal that the lift of an angle of a tile has the same measure, except at a ramification point where the angle is divided by $e(p, z_0)$. The pullback of a tile allows to prove the following version of the Riemann-Hurwitz theorem.

Proposition 6.21 *Let $p : S \to T$ be a branched cover of degree d, branched over t*

points $w_1, \ldots, w_t$, and let σ and τ be the genus of S an T respectively. Then,

$$2 - 2\sigma = d(2 - 2\tau) - \sum_{z \in S} (e(p, z) - 1)$$

or

$$2 - 2\sigma = d(2 - 2\tau - t) + \sum_{j=1}^{t} \left| p^{-1}(w_j) \right|$$

Proof. Let V_T, E_T, F_T be the number of vertices, edges and tiles of tiling $\mathcal{T}$ on T, with branch points at vertices, and V_S, E_S, F_S number of vertices, edges and tiles of the pullback $p^{-1}(\mathcal{T})$. We calculate the Euler characteristic of S in two ways:
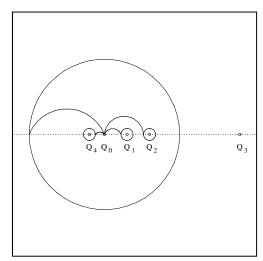
$$
\begin{aligned}
2 - 2\sigma &= V_S - E_S + F_S \\
&= dV_T - dE_T + dF_T - \text{overcount} \\
&= d(2 - 2\tau) - \text{overcount},
\end{aligned}
$$

where overcount accounts for the ramification at vertices of S. Now, as p is a covering space over the interior of the edges and the tiles there are d edges or tiles lying above each edge or tile of T, thereby accounting for the term $dE_T + dF_T$. Next, let w be a vertex in $\mathcal{T}$ and let $z_1, \ldots, z_k$ be the points lying above Q. The number contributed to the overcount is

$$
\begin{aligned}
d - k &= \sum_{z \in p^{-1}(w)} e(p, z) - \sum_{z \in p^{-1}(w)} 1 \\
&= \sum_{z \in p^{-1}(w)} (e(p, z) - 1).
\end{aligned}
$$

Thus the total overcount is $\sum_{z \in S} (e(p, z) - 1)$. This sum is finite since the terms vanish at unramified points. The formula is now established. ∎

From now on we are going to assume that $T = S^2 = \widehat{\mathbb{C}}$. Pick a non-branch point $w_0 \in T$ (normally $w_0 = 0$) and let $w_1, \ldots w_k$ be the branch points of p, ordered counter-clockwise around w_0. In T° let γ_i be constructed from a path ζ_i that proceeds directly from z_0 to a point y_i near z_i, makes a small counter-clockwise loop ϵ_i around z_i and returns to w_0 backwards along ζ_i, so that $\gamma_i = \zeta_i * \epsilon_i * \zeta_i^{-1}$. If some $w_i = \infty$ then we replace ϵ_i with a large circle containing all the branch points. The loops are only supposed to intersect at w_0. Standard facts are that $\pi_0(T^\circ, w_0) = \langle \gamma_1, \ldots, \gamma_t \rangle$, that $\gamma_1 * \cdots * \gamma_t$ null-homotopic, and that $\gamma_1 * \cdots * \gamma_t = 1$ is the only relation. Such a system of loops is called a marking. Two markings relevant to quadrilateral tilings are shown in Figures 6.2 and 6.3. In these figures Q_i represent w_i, Q_3 is the point at infinity and a counterclockwise circle around ∞ it is a large clockwise circle. The dotted line is a symmetry line for the branch points which is relevant in the case of tilings.
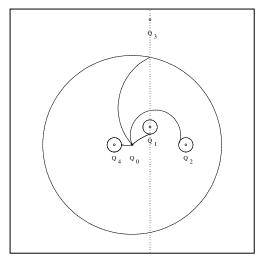
Fig 6.2. Marking for real points   Fig 6.3. Marking for symmetric points

Let us compute the monodromy operator, using a marking. According to Remark 6.8 $\mu_{\gamma_i} = \kappa\mu_{\epsilon_i}\kappa^{-1}$, where κ is a bijection from $p^{-1}(w_0)$ to $p^{-1}(y_i)$. Thus that the cycle structure of μ_{γ_i} is determined by μ_{ϵ_i}. Over ϵ_i we get the following behaviour. Assume that ϵ_i bounds a disc W surrounding w_i such that there is a $z_j \in p^{-1}(w_i)$, and V_j surrounding z_j such that in local coordinates p maps V_j to W_j by $z \to z^{e_j}$. Assume that the ϵ_i is chosen small enough so that $z \to z^{e_j}$ is valid for each V_j. As a full circle is traced out on ϵ_i, the lift moves through $1/e_j$ of the full circle boundary of V_j. Going around ϵ_i has a lift going through $2/e_j$ of the full circle, and so on. It follows then that $p^{-1}(y_i) \cap V_j$ are e_j points equally spaced around ∂V_j and that μ_{ϵ_i} cyclical permutes them. Thus μ_{ϵ_i} is an e_j-cycle when acting on $p^{-1}(y_i) \cap V_j$. Taking other V_j into account we see that μ_{ϵ_i} and hence μ_{γ_i} is a product of the form e_1-cycle $\times$ e_2-cycle $\times \cdots \times$ e_k-cycle. We are now ready to state Riemann's Existence theorem.

Theorem 6.22 *Let S be Riemann surfaces of genus σ and let $T = S^2$ be the sphere. Then there is a branched covering $p : S \to T$ of degree d with t ramified fibres of types $e_{1,1}^{n_{1,1}} \cdots e_{k_1,1}^{n_{1,k_1}}, \ldots, e_{t,1}^{n_{t,1}} \cdots e_{t,k_t}^{n_{t,k_t}}$, respectively, if and only if*

$$2 - 2\sigma = 2d - \sum_{i=1}^{t}\sum_{j=1}^{k_i} n_{i.j}(e_{i,j} - 1), \text{ or} \qquad (6.8)$$

$$2 - 2\sigma = d(2 - t) + \sum_{i=1}^{t}\sum_{j=1}^{k_i} n_{i.j},$$

and there are permutations $\mu_1, \ldots, \mu_t \in \Sigma_d$, such that

1. *μ_i has cycle type $e_{i,1}^{n_{i,1}} \cdots e_{i,k_i}^{n_{i,k_i}}$*

2. *$\mu_1 \cdots \mu_t = 1$, and*

3. $\mu_1, \ldots, \mu_t$ generate a transitive subgroup M of Σ_d.

Proof. We have sketched most of the ideas of the proof in the discussion leading up to the statement of the theorem. At the end of this section we give an more explicit geometric construction of the covering space using the original ideas of sheets of a Riemann surface. The equation 6.8 is the Riemann Hurwitz equation of the cover. In statement 2 we are assuming that the multiplication in Σ_d as in Remark 6.7 so that the monodromy representation is a homomorphism. The relation in statement 2 then follows from $\gamma_1 * \cdots * \gamma_k = 1$. Since $\pi_0(T^\circ, w_0) = \langle \gamma_1, \ldots, \gamma_t \rangle$ then $\mu_1, \ldots, \mu_t$ generate the image of the monodromy representation which is a transitive subgroup of Σ_d, by Proposition 6.12.

Now suppose that we have cycles $\mu_1, \ldots, \mu_t$ satisfying 1, 2, and 3. Let T° be the sphere with t punctures as described above. Then there is a permutation representation $\pi_0(T^\circ, w_0) \to \Sigma_d$ defined by $\gamma_i \to \mu_i$, since $\gamma_1 * \cdots * \gamma_t = 1$ is the only relation among the $\gamma_1, \ldots, \gamma_t$ and $\mu_1 \cdots \mu_t = 1$. Since the representation is transitive and there is a covering space of T° with the prescribed monodromy. If V_i° is a small punctured disc at w_i then $p^{-1}(V_i^\circ)$ is a disjoint union of punctured discs $U_{i,j}^\circ$ such that the map $p : U_{i,j}^\circ \to V_i^\circ$ is given by $z \to z^{e_j}$ and the sequence of e_j's is determined by the cycle type of μ_i. Now all the punctures may be filled in and the map extended so that S and T are closed surfaces and the fibre type of $p^{-1}(w_i)$ is $e_{i,1}^{n_{i,1}} \cdots e_{i,k_i}^{n_{i,k_i}}$. Finally the formula (6.8) shows that the genus S is correct. ∎

Remark 6.10 If there are two sets of permutations $\mu_1', \ldots, \mu_t'$ and $\mu_1, \ldots, \mu_t$, Then the branched covers are conformally equivalent if there is a permutation $\xi \in \Sigma_d$ such that $\mu_i' = \xi^{-1} \mu_i \xi$, for all i. This amounts to relabelling.

Remark 6.11 If $p : S \to T$ a Galois cover and $z_0 \in p^{-1}(w_0)$ is fixed there is a homomorphism $\pi_0(T^\circ, w_0) \to G = \mathrm{Gal}(S/T)$, by Proposition 6.17. Let $c_i = \eta(\gamma_i)$. Then $(c_1, \ldots, c_t)$ is a generating vector for the action of G on S as in the discussion in Chapter 2.

Example 6.9 The map $z \to z^3 - 3z$ of the sphere to itself was discussed in Example 6.8. The ramified fibre types are $1 \cdot 2$, $1 \cdot 2$, and 3. Up to conjugacy the only choice for the monodromy is $\mu_1 = (1, 2)$, $\mu_2 = (2, 3)$, and $\mu_3 = (1, 2, 3)$.

Construction of Galois branched covers Given a branched cover $p : S \to T$, we get a connected unbranched cover $p : S^\circ \to T^\circ$. Now we may extend this to an unbranched Galois cover $U^\circ \xrightarrow{q} S^\circ \xrightarrow{p} T^\circ$ and then a branched Galois cover $U \xrightarrow{q} S \xrightarrow{p} T$. We may represent this in diagram form:

$$
\begin{array}{ccc}
U & \xrightarrow{q} & S \\
 & \searrow^{r} & \downarrow^{p} \\
 & & T
\end{array}
\tag{9}
$$

Suppose that we also know the structure of the monodromy representation as in Proposition 6.22. We would like to compute $\mathrm{Gal}(U/T)$, and a monodromy representation for the cover $r : U \to T$. According to Proposition 6.19 $\mathrm{Gal}(U/T)$ is isomorphic to the image $M = \langle \mu_1, \ldots, \mu_t \rangle$ of the monodromy representation of $p : S \to T$. Now we may label the points of $p^{-1}(w_0)$ as $z_1, \ldots, z_d$ so that the monodromy representation acts by

$$z_j^{\gamma_i} = z_j \cdot \mu_j = z_{j \cdot \mu_i}$$

recalling that the μ_i's act on $\{1, \ldots, d\}$ on the right. We also know that U is a Galois cover of S with $\mathrm{Gal}(U/S)$ corresponding to $H = \{\mu \in M : 1 \cdot \mu = 1\}$. For notational convenience let $G = \mathrm{Gal}(U/T)$ and $H = \mathrm{Gal}(U/S)$, in our continuing discussion.

Example 6.10   Continuing Example 6.8, We have $G \simeq \Sigma_3$, and H is identified with $\langle (2,3) \rangle$ under this isomorphism.

Next we calculate the monodromy representation of a Galois branched cover and then show how to construct the surface from the monodromy representation. To this end recall from Proposition 6.17 that there is a map $\eta : \pi_0(T^\circ, w_0) \to G$ such that

$$u_0^\alpha = \eta(\alpha)u_0,$$

for a fixed $u0 \in r^{-1}(w_0)$ and arbitrary $\alpha \in \pi_0(T^\circ, w_0)$. Let $c_i = \eta(\gamma_i)$. Because of the commuting diagram of covers we get the following relation between the monodromies of $S \xrightarrow{p} T$ and $U \xrightarrow{r} T$

$$q(u^\alpha) = q(u)^\alpha.$$

Since the map $q(u) \to q(u)^\alpha$, is the permutation μ_α, then the isomorphism between G and M is given by $\eta(\alpha) \leftrightarrow \mu_\alpha$, in particular $c_i \leftrightarrow \mu_i$. Now assume choices have been made so that $q(u_0) = z_1$ and represent the elements of $r(w_0)$ by $g_1 u_0, \ldots, g_k u_0$ where $g_1, \ldots, g_k$ is an enumeration of the elements of G. Let g be an arbitrary element of G, let $\alpha, \alpha_1, \ldots, \alpha_k$ be paths satisfying $g = \eta(\alpha)$, $g_1 = \eta(\alpha_1), \ldots, g_k = \eta(\alpha_k)$, and let $\nu = \mu_\alpha, \nu_1 = \mu_{\alpha_1}, \ldots, \nu_k = \mu_{\alpha_k}$. Then,

$$(g_j u_0)^\alpha = g_j(u_0^\alpha) = g_j g u_0.$$

Also

$$
\begin{aligned}
q(g_j g u_0) &= q(u_0^{\alpha_j * \alpha}) \\
&= (q(u_0))^{\alpha_j * \alpha} \\
&= z_1^{\alpha_j * \alpha} \\
&= z_1 \nu_j \nu.
\end{aligned}
$$

Thus the monodromy action of $\pi_0(T^\circ, w_0)$ is given by the right action of G on itself which in turn is equivalent to the right action of M on itself. Therefore, to compute the monodromy action of $\pi_0(T^\circ, w_0)$ concretely in terms of M we enumerate the elements of M as $\nu_1, \ldots, \nu_k$ and the monodromy operator determined by γ_i is the permutation induced by the right action $\nu_j \to \nu_j \mu_i$. However, note that the Galois group action on the fibre, $x \to c_i x$, is given by the left action $\nu_j \to \mu_i \nu_j$ for the Galois action is $g_j u_0 \to g g_j u_0$, and $q(g_j g u_0) = z_1 \nu_j \nu$.

We formalize the forgoing in a theorem.

Theorem 6.23 *Let $p : S \to T$ be a branched cover induced by the permutations $\mu_1, \ldots, \mu_t$ satisfying the requirements of Theorem 6.22, and let M be the group generated by the μ_i. Let $\nu_1, \ldots \nu_k$ be an enumeration of the elements of M, and let M_i denote the permutation of $\nu_1, \ldots \nu_k$ induced by $\nu_j \to \nu_j\mu_i$. Then the Galois cover $r : U \to T$, induced by $p : S \to T$, as in 6.9, is induced by the permutations M_i per the Riemann Existence Theorem 6.22. The action of the Galois group on the fibre is given by the left action of M on M, with the generating vector $(c_1, \ldots, c_t) = (\eta(\gamma_1), \ldots, \eta(\gamma_t))$ given by $\nu_j \to \mu_i\nu_j$. In particular the Galois action commutes with the monodromy action.*

Relation to field theory.   Let S be a surface and let $f_1 : S \to \widehat{\mathbb{C}}$, $f_2 : S \to \widehat{\mathbb{C}}$, be two holomorphic mappings that are either a branched covering or a constant map with a finite value. It may be easily shown that the maps defined by af_1, $a \in \mathbb{C}$, $f_1 + f_2, f_1 - f_2, f_1f_2, f_1/f_2, f_2 \neq 0$ are all holomorphic maps. Thus the set of such functions forms a field, containing a copy of the complex numbers. This field is called the field of meromorphic functions on S and is denoted $\mathbb{C}(S)$. If $p : S \to T$ is a branched covering of degree d, then the transformation $f \to f \circ p$ gives an embedding of the fields $p^* : \mathbb{C}(T) \to \mathbb{C}(S)$. The field extension is finite, algebraic of degree d. On the other hand if K is a finite algebraic extension of $\mathbb{C}(T)$, then there is a cover $p : S \to T$ such that the inclusion $\mathbb{C}(T) \to \mathbb{C}(S)$ is equivalent to $\mathbb{C}(T) \to K$. If $h \in \mathrm{Gal}(S/T)$, then h^* is an automorphism of $\mathbb{C}(S)$ satisfying $h^*p^* = p^*$ since $ph = p$. It follows that h^* fixes the subfield $p^*(\mathbb{C}(T))$ and hence $h^* \in \mathrm{Gal}(\mathbb{C}(S)/ \, p^*(\mathbb{C}(T)))$, computed as a Galois group of field extensions. For notational convenience we drop the p^* and write $\mathrm{Gal}(\mathbb{C}(S)/\mathbb{C}(T))$. The main point is that the map $\mathrm{Gal}(S/T) \to \mathrm{Gal}(\mathbb{C}(S)/\mathbb{C}(T))$, given by $h \to h^*$ is an isomorphism of Galois groups interrelating the two different versions of the Galois correspondence.

6.5.1 Addendum: Sheet construction of branched covers.

Select a marking of t loops based at w_0 on the sphere as described as above. Pick a point w not on the marking a draw a system of t non-intersecting smooth paths $\delta_1, \ldots, \delta_t$ that emanate from w in counterclockwise order and that do not cross the marking except that δ_i terminates at a point on the circle of γ_i.. Cut out the disks encircled by the marking and also cut this punctured sphere along the paths δ_i. Let T° denote the sphere with the discs in the marking cut out but not cut along the δ_i. What is left over may be compactified and flattened out to a $2t$-sided polygon in which every second vertex has been clipped off (because we removed the discs) and the other vertices all come from the base point w. As we go around the polygon in a clockwise fashion the first two sides will trace out a path consisting of δ_1^+, a circle κ_1 disconnected at one point and the path δ_1^-. On the sphere, δ_1^+ and δ_1^- correspond to the same path traversed δ_1 in opposite directions. The path κ_1 corresponds to travelling around the circle of γ_1 in the direction induced by the orientation of γ_1. The next two sides will trace out $\delta_2^+, \kappa_2, \delta_2^-$, and so on. The interior of this polygon will contain our marking except that each circle will have been cut at one point. The

arcs of the marking will emanate from base w_0 in the interior of the polygon and intersect the boundary in exactly one point on a κ_i. We will call the polygon we constructed a sheet. Now if n is the degree of our monodromy representation, make n copies of our sheet and denote the i'th copy of w_0 by w_i. We are going to attach the sheets together according to a recipe derived from the monodromy from which we will then create a covering of T°, which may be completed to a branched covering of sphere by sewing discs back in.

Now suppose $\mu : \pi_1(T^\circ, w_0) \to \Sigma_n$ a monodromy or permutation representation of degree n. Pick γ_i and compute the permutation $\pi = \mu(\gamma_i)$ we join sheet j and sheet k along δ_i^- side of sheet j and the δ_i^+ of sheet k if $j^\pi = k$. Now by our construction every point not on κ_i the sheet fitting together to form a small euclidean neighbourhood, either by two sheets fitting together along δ-curves or .t sheets fitting together at the vertex point defined by w. The κ-curves are fit together *to* form boundary circles. Let us call S° the surface with boundary so formed, by construction there is a canonical map $S^\circ \to T^\circ$. The monodromy map of $S^\circ \to T^\circ$ is correct suppose that we try lifting γ_i at w_j the centre of sheet j. The lifted path start at w_j the centre of sheet j and heads for the centre of the boundary arc κ_i. By construction it crosses into sheet k by crossing the δ_i border, continues along κ_i and then heads in toward the centre w_k sheet k. Thus $w_j^{\gamma_i} = w_k$ as prescribed by the original monodromy. The remaining task is to fill in the holes with discs and extend the map $S \to T$ to a branched covering with appropriate branching orders. This is a straightforward task left to the reader.

6.6 The dual tiling and the path fundamental group

First we give a very explicit definition of the path fundamental group for kaleidoscopic tilings. In this way the isomorphism of the path fundamental group and the Galois fundamental group can be explicitly constructed.

Kaleidoscopic tilings Given a kaleidoscopic tiling by triangles, we can construct a dual tiling as follows. Let I_g be the incentre of the tile $\Delta_g = g\Delta_0$. This is the centre of the circle which is tangent to all three sides of the triangle, or alternatively the intersection of the three angle bisectors of the triangle. Note that the distance from the incentre to each of the sides is the same. Given this geometrically invariant definition, it is obvious that G^* permutes these incentres, i.e., $gI_h = I_{gh}$. If s is a reflection in the side of Δ_g then connect I_g and I_{sg} by a line segment. Now I_g and I_{sg} are s-mirror images of each other across the common edge $e = \Delta_g \cap \Delta_{sg}$. Thus the line segment joining I_g and I_{sg} is perpendicularly bisected by e, and s maps this segment onto itself, turning it end for end. The dual tiling has the following vertices, edges and tiles. The vertices are the $2|G|$ incentres, the edges are the $3|G|$ edges connecting the incentres constructed as above. Note that all the edges of this tiling have the same length. The tiles are regular $2l$, $2m$ or $2n$ polygons centred at the vertices of the original tiling.

The dual graph is the collection of dual vertices and edges, we do not need the dual faces to construct the path fundamental group. Let I_0 be the incentre of the master

tile. Consider the collection of all closed loops in the dual graph starting and ending at I_0. Each such loop is a concatenation of edges from the dual tiling of the form $f_1 f_2 \cdots f_n$ where f_1 starts at I_0 the initial point of f_{i+1} is the end point of f_i, and f_n ends at I_0. Two paths are homotopic if one can be continuously deformed into the other, keeping the base point I_0 fixed throughout the deformation. We denote the set of homotopy classes of closed paths based at I_0 in the dual graph by $\pi_0(S, I_0)$ and call it the *(path) fundamental group* of S based at I_0. Two paths are multiplied by concatenating paths.

We now show how $\pi_0(S, I_0)$ is canonically isomorphic to the fundamental group Γ defined above. For a path $f_1 f_2 \cdots f_n$ in $\pi_0(S, I_0)$ let $e_1, e_2, \ldots, e_n$ be the sequence of edges in the kaleidoscopic tiling crossed by the path. Namely, e_i is the edge of the kaleidoscopic tiling that meets f_i at a right angle. The universal covering has a dual tiling, let $\widetilde{f_1} \widetilde{f_2} \cdots \widetilde{f_n}$ be the path in this dual tiling that covers $f_1 f_2 \cdots f_n$. The lift starts at the incentre $\widetilde{I_0}$ of the master tile $\widetilde{\Delta_0}$ in the universal cover. Let $\widetilde{e_1}, \widetilde{e_2}, \ldots, \widetilde{e_n}$ be the sequence of edges in the kaleidoscopic tiling in the universal cover such that $\widetilde{f_i}$ meets $\widetilde{e_i}$. We then define $\zeta : \pi_0(S, I_0) \to \Gamma$ by

$$\zeta(f_1 f_2 \cdots f_n) = r_{\widetilde{e_n}} r_{\widetilde{e_{n-1}}} \cdots r_{\widetilde{e_1}}.$$

Using the idea of a reflective walk we can determine ζ images with out lifting paths. Let $\{\widetilde{p}, \widetilde{q}, \widetilde{r}\}$ be the appropriately named, set of reflections in the edges of the master tile $\widetilde{\Delta_0}$. Let $\widetilde{r_i} = \widetilde{p}, \widetilde{q}$, or $\widetilde{r}$ according to the type of f_i. The dual edge f_i has type p if the edge it crosses, e_i, has type p, and similar definitions hold for q-type and r-type. Then we have

$$\zeta(f_1 f_2 \cdots f_n) = \widetilde{r_1} \widetilde{r_2} \cdots \widetilde{r_2}.$$

Homotopic paths the same image in Γ, indeed if $g = \zeta(f_1 f_2 \cdots f_n)$. Then $\widetilde{f_1} \widetilde{f_2} \cdots \widetilde{f_n}$ is a path from $\widetilde{I_0}$ to $g\widetilde{I_0}$. Two paths are homotopic if and only if their lifts have the same endpoint or equivalently if and only if they have the same image in Γ. Thus ζ is an isomorphism.

General tilings   Now let S (possibly an unbounded surface such as the euclidean or hyperbolic plane) have a general tiling by polygons. The polygons will be simply-connected regions on the surface, bounded by a sequence of edges. The edges are simple, smooth curves, that do not close up. Two adjacent edges meet at a nonzero angle. We not assuming any geometrical information beyond differentiability. We assume that if two polygons meet in part of an edge then their intersection includes the entire edge including the endpoints. We select a distinguished tile Δ_0 of the tiling which we still call the master tile.

Now construct the dual tiling as follows. In each polygon Δ of the original tiling select a (dual) vertex P' in the interior of the polygon. If two polygons Δ_1 and Δ_2 meet along an edge e then join the dual vertices $P_1' \in \Delta_1$ and $P_2' \in \Delta_2$ by a simple smooth curve e' that intersects e in a single point. Around each vertex P of the original tiling, consider the edges emanating from the vertex. Each of these edges meets a unique dual edge. These dual edges form the boundary of a curvilinear polygon Δ' on S that contains P. Thus we have constructed a tiling on S such that each face Δ, edge e

or vertex P is incident with a unique vertex P', edge e' or face Δ' of the dual tiling. The dual graph is the collection of dual vertices and edges, we do not need the dual faces.

The path fundamental group is defined as in the kaleidoscopic case except for the following notational changes. Let x_0 be the chosen point of the master tile Δ_0. Again we denote the set of homotopy classes of closed paths based at x_0 in the dual graph by $\pi_0(S, x_0)$ and call it the *(path) fundamental group* of S based at x_0.

6.7 Non-kaleidoscopic tilings

Suppose we have a tiling of an orientable surface S by congruent polygonal tiles such that the angle at each vertex is an integer submultiple of π. We will not assume that the tiling is kaleidoscopic but merely that:

- any two tiles meeting along an edge the tiles are mirror images of each other in the edge,

- tiles are arranged around the vertex with the appropriate dihedral symmetry,

- the tiles have been colored black and white so that along any edge we have a black and white tile meeting, and

- the edges of each tile have been given distinct labels so that tiles that meet along an edge of the same type and this labelling is preserved by the local reflections.

Note that the assumptions imply that no tile meets itself along an edge or at a vertex. We are not assuming that the local mirror images extend to reflections of the entire surface. Now we shall try to construct the surface as a quotient of the universal covering space and at the same time relate it to the fundamental group as defined in the last section. Select a tile $\overline{\Delta_0}$ in S and a congruent tile Δ_0 in the universal cover X. Let $\mathfrak{R}$ be set of the reflections in the sides of Δ_0 and let Λ^* be the group generated by $\mathfrak{R}$. We may use $\mathfrak{R}$ as the labeling set for the edges of the tiling of S by labeling the edges of Δ_0 in the obvious fashion and transferring that labeling to $\overline{\Delta_0}$ through the given congruence. Now the rest of the labelling on the edges in S is forced by the compatibility assumptions above. The set of tiles $\Lambda\Delta_0 = \{g\Delta_0 : g \in \Lambda\}$ is a tiling of X.

Define a subgroup Γ of Λ^* as follows. Let $x_0 \in \Delta_0$ be a distinguished point, let $\alpha(t), 0 \leq t \leq 1$ be a loop starting and finishing at x_0. By deforming the loop slightly we may assume that it does not pass through any of the vertices of the tiling. As we follow the path along we will enter a series of tiles $\overline{\Delta_1}, \ldots, \overline{\Delta_n} = \overline{\Delta_0}$, suppose we enter tile $\overline{\Delta_k}$ through an edge of type $r_k \in \mathfrak{R}$. As we move along the path we define a sequence of tiles in the universal cover by $\Delta_k = r_1 r_2 \cdots r_k \Delta_0$. By construction, Δ_k covers $\overline{\Delta_k}$ as the notation suggests. We include $r_1 r_2 \cdots r_n$ in Γ for every possible closed loop. Since we were able to consistently color tiles then the sequence of colors starting with $\overline{\Delta_1}$ is: $\overline{\Delta_1}$ black, $\overline{\Delta_1}$ white, $\overline{\Delta_2}$ black, $\overline{\Delta_n} = \overline{\Delta_0}$, white. It follows that n is even and hence that $g = r_1 r_2 \cdots r_n \in \Lambda$. Collect all such elements into a set Γ.

Now Γ is a subgroup of Λ as follows. Suppose that g and h correspond to the paths α and β. Then gh corresponds to the path $\alpha * \beta$ which is the path formed by travelling first along α and then along β. The element g^{-1} is obtained by travelling backwards along α.

Example 6.11   The process is easily visualized by lifting a pattern of squares on a torus to the standard tiling of the plane by unit squares. Pick a rectangular array of squares in the plane with an even number of squares in each direction, say $2m \times 2n$. Now form a torus by identifying top edge to bottom edge and left edge to right edge. Because both directions have an even number of squares the tiling on the torus satisfies the compatibility conditions. The fundamental group is given by $\Gamma = 2m\mathbb{Z} \times 2n\mathbb{Z}$.

Proposition 6.24   *Let the tiling S be given as above and let the $\Gamma \subseteq \Lambda$ be the pair of groups constructed above. Then Γ is a fixed point free group of orientation preserving isometries and there is an isometry $h : X/\Gamma \to S$ such that the tiling induced on X/Γ from the tiling on X is mapped onto the original tiling of S.*

Proof.   Since Γ is orientation preserving and fixed point free X/Γ is a surface. Since Γ permutes the tiles on X isometrically an induced tiling on X/Γ is easily constructed on X/Γ by taking the images of tiles. The composed map $h \circ q : X \to X/\Gamma \to S$, and hence h, are fairly easy to construct. The only thing not straight forward is that Γ acts freely. If Γ is not fixed point free then the stabilizer Γ_v of some vertex v of the tiling on X is nontrivial. But this is impossible since the number of polygons meeting at v and at $h(q(v))$ is the same by construction.

By our previous definitions, Γ is the fundamental group of S. The above construction gives a direct link to the standard definition of the fundamental group, constructed through the tiling.

Corollary 6.25   *Every non-kaleidoscopic tiling satisfying the compatibility conditions may be constructed by selecting a fixed point free subgroup of the universal tiling group Λ^* and constructing the quotient tiling on X/Γ. The tiling is kaleidoscopic if and only if $\Gamma \triangleleft \Lambda^*$. The tiling groups of S are given by:*

$$G^* = \Lambda^*/\Gamma, \ G = \Lambda/\Gamma.$$

6.7.1 Covering of tilings

Definition 6.12   Let $p : S \to T$ be a covering space of surfaces. Suppose that both S and T have tilings such that each tile of S is mapped isometrically onto a tile of T. Then the map p is called a covering of tilings.

Proposition 6.26   *Every covering of tiling $p : S \to T$ occurs in the following way: There is a tiling on the universal cover of X with tiling group Λ^* and fixed point free*

subgroups of $\Gamma_S \subseteq \Gamma_T \subseteq \Lambda$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
X & \stackrel{id}{\to} & X \\
\downarrow & & \downarrow \\
S & \leftrightarrow & X/\Gamma_S \\
\downarrow p & & \downarrow \\
T & \leftrightarrow & X/\Gamma_T
\end{array}
$$

The unlabeled horizontal arrows are homeomorphisms, the unlabeled vertical arrows are natural covering space projections, and Γ_S and Γ_T are the fundamental groups of S and T respectively.

Lifting tilings via monodromy   Now suppose that we have a specification by monodromy for a branched cover as in Theorem 6.22. We wish to construct a simple tiling on S by pulling back a simple tiling on T. For the sake of argument we are going to assume that the branch points $w_1, \ldots, w_t$ are equally spaced along the equator. Our tiling on T will have two tiles, Δ_- and Δ_+ the northern and the southern hemispheres, the edges e_i will be the portions of the equator lying between two adjacent vertices, w_i and w_{i+1} (indices $\bmod\, t$), and of course the vertices are the branch points w_i. The base point is $w_0 = 0$. When represented in the plane Δ_+ is the unit disc, Δ_- is the exterior of the unit disc and the e_i and w_i are arcs and edges on the unit circle. In the branched cover we will have the following:

open tiles	Δ_+^j, Δ_-^j, $j = 1, \ldots, d$, let w_0^j be the preimage of w_0 in Δ_+^j
open edges	e_i^j, $i = 1, \ldots, t$, $j = 1, \ldots, d$
vertices	w_i^j, $i = 1, \ldots, t$, $j = 1, \ldots, d$, with some identifications

We need to describe how the various tiles edges and vertices are glued together in the covering space. Glue the edge e_i^j to tile Δ_+^j at the location where the i'th edge should go, guided by the homeomorphism $p : \Delta_+^j \to \Delta_+$. Next glue Δ_-^j to the closed tile Δ_+^j along edge e_t^j. Now we are going glue Δ_-^j to $\Delta_+^{j_1}$ along the e_1 edges, the assignment $j \to j'$ will be determined by the monodromy. Consider lifting the loop γ_1, It heads towards w_1 enters Δ_- across e_t, reenters Δ_+ across e_1 and heads back to w_0. Now consider the lift $\widetilde{\gamma}_1$ starting at w_0^j It crosses into Δ_-^j across e_t^j edge and then crosses the e_1 type edge of Δ_-^j into $\Delta_+^{j_1}$ stopping at $w_0^{j_1}$. Since the monodromy has been specified $j_1 = j^{\gamma_1} = j\mu_1$. Thus tile Δ_-^j and $\Delta_+^{j\mu_1}$ are joined along their e_1 type edge which is the edge $e_1^{j_1} = e_1^{j\mu_1}$. Now consider lifting $\gamma_1 * \gamma_2$. The path $\gamma_1 * \gamma_2$ follows γ_1 as before and then heads towards w_2, enters Δ_- across e_1, returns to Δ_+ across e_2 and heads back to w_0. The lift $\widetilde{\gamma_1 * \gamma_2}$ will follow $\widetilde{\gamma}_1$ to $w_0^{j_1}$ then reenter Δ_-^j across $e_1^{j_1}$ and then enter $\Delta_+^{j_2}$ and travel to $w_0^{j_2}$. Obviously $j_2 = j_1\mu_2 = j\mu_1\mu_2$ and we glue Δ_-^j to $\Delta_+^{j_2}$ along $e_2^{j_2}$. Proceeding inductively we define $j_k = j\mu_1 \cdots \mu_k$ and join Δ_-^j to $\Delta_+^{j_k}$ along $e_1^{j_k}$. Note that as $\mu_1 \cdots \mu_t = 1$ the joining rules along e_t edges are unambiguously defined. We have now defined all the identifications for tiles and edges, we need to describe where to put the vertices. If we follow γ_i repeatedly we

see that the Δ_+^j tiles, for j in a γ_i-orbit are all joined at the w_i vertex. Therefore, we identify the w_i^j according to these orbits and paste them in to fill the punctures. If the covering is a Galois covering then, since the Galois action commutes with the monodromy action, the Galois action on the tiling preserves the identifications we have made.

6.7.2 Symmetry groups of non-kaleidoscopic tilings

If we have a non-kaleidoscopic tiling not all edges determine a reflection though there may be some that do. So let us determine the portion of the symmetries of such a tiling that come from the symmetry group. Here is the statement.

Proposition 6.27 *Let S be a surface with a non-kaleidoscopic tiling induced by the pair $\Gamma \lhd \Lambda^*$ in the universal cover X. Then the symmetries of the tiling induced by elements of Λ^* is isomorphic to $N_{\Lambda^*}(\Gamma)/\Gamma$.*

Proof. Suppose that $g \in \Lambda^*$ determines a tiling automorphism $\overline{g}$ of S. Then we have the following diagram

$$
\begin{array}{ccc}
X & \xrightarrow{g} & X \\
\downarrow p & & \downarrow p \\
S & \xrightarrow{\overline{g}} & S
\end{array}
$$

i.e., $L_{\overline{g}} \circ p = p \circ L_g$ where L_g, $L_{\overline{g}}$ denote the mappings of left action by g and $\overline{g}$. Now take the full inverse image via both maps of the point $\overline{g}p(x_0) = p(gx_0)$. We get

$$
\Gamma x_0 = p^{-1}(p(x_0))) = p^{-1} \circ L_{\overline{g}}^{-1}(\overline{g}p(x_0)) = L_{\overline{g}}^{-1} \circ p^{-1}(p(gx_0)) = L_{\overline{g}}^{-1}(\Gamma gx_0) = g^{-1}\Gamma gx_0.
$$

It follows that $\Gamma = g^{-1}\Gamma g$ since that the action of Γ is fixed point free. It follows that g must normalize Γ. A similar argument show that if g normalizes Γ then the map $\overline{g}$ is defined.

6.7.3 Some non-kaleidoscopic Hurwitz tilings

Here we produce a family of non-kaleidoscopic tiling by $(2,3,7)$ triangles. Let $T_{l,m,n}^*$ denote the tiling group generated by an (l,m,n) triangle, and let $\Gamma \to T_{l,m,n}^* \xrightarrow{\eta} G^*$ be the exact sequence determined by a tiling of a surface S with tiling group G^*. Now suppose that we can find a subgroup H of G which has no fixed points on S. Then S/H should carry an (l,m,n)-tiling that will be kaleidoscopic if and only if H is normal. An alternative way to see this is to let Δ be the inverse image of H under q. If $g \in \Delta$ has a fixes $x \in \mathbb{H}$ then $\eta(g)$ fixes $\overline{x}$ on S, a contradiction. Thus $\mathbb{H}/\gneqq = S/H$ has an (l,m,n) tiling.

Now the Hurwitz examples. If p is a prime satisfying $p^2 \equiv 1 \bmod 7$ then $G = PSL_2(p)$ has a $(2,3,7)$-action on a surface S of genus $\sigma = 1 + \frac{p(p^2-1)}{168}$. If an element of G fixes a point x on S then it has order $2, 3$ or 7, so lets find a large subgroup not divisible by any of these numbers. The group G contains a maximal subgroup B of order $\frac{(p-1)p}{2}$ consisting of upper triangular matrices $B = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a, b \in \mathbb{F}_p, a \neq 0 \right\} /\{\pm I\}$. Now the

subgroups $D = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a \in \mathbb{F}_p^* \right\} / \{\pm I\}$, and $P = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : a, b \in \mathbb{F}_p, a \neq 0 \right\}$
are cyclic of order $\frac{p-1}{2}$ and p respectively, and the order of elements of B are p or a
divisor of $\frac{p-1}{2}$. Write $\frac{p-1}{2} = qq'$ where q and q' relatively prime and prime factors of q'
are contained in $\{2, 3, 7\}$. Correspondingly $D = E \times E'$ where $|E| = q$ and $|E'| = q'$.
The subgroup EP has order qp and no elements of order $2, 3$ or 7. Thus S/EP has a
Hurwitz tiling with $\frac{2|G|}{|EP|} = 2(p+1)q'$ triangles. Since $\sigma - 1 = (\sigma' - 1)|EP|$ then the
genus σ' equals $1 + \frac{(\sigma-1)}{|EP|} = 1 + \frac{2|G|}{84|EP|} = 1 + \frac{(p+1)q'}{42}$ Here are the first five cases:

p	13	29	41	43	71
q'	6	14	4	21	7
$|G^*| = p(p^2 - 1)$	2184	24360	68880	79464	357840
# tiles on S/H $(= 2(p+1)q')$	168	420	336	1848	1008
σ	27	291	821	947	4261
σ'	3	11	5	23	13

References to original sources for some of the facts about $PSL_2(p)$ can be found in
the Glover-Sjerve paper [16] and the paper on Hurwitz action by Bujalance et. al.
[12].