7. Homology and Cohomology of Surfaces

7.1 Introduction to Homology - General coefficients

In this section we give very abbreviated discussion on homology and cohomology and its relation to tilings and covers. The homology groups give an algebraic handle for discussing separability of symmetries and coverings of tilings. The cohomology groups are useful in discussing Hecke rings in the next chapter. The construction of homology and cohomology are also motivated by trying to understand line integrals and Stokes and Green's theorem on a surface. In particular the theorem

$$\iint\limits_{W} d\omega = \int\limits_{\partial W} \omega$$

for differential 1-forms is a key idea and result.

Let S have a tiling by polygons. We do not require that the tiling be a kaleidoscopic tiling by triangles, simply that two polygons meet either in a vertex or an edge which is a complete edge of both polygons. A polygon or edge is the homeomorphic image of a polygon or edge, respectively, on the universal cover. The edges are simple, smooth curves, that do not close up. two edges meet at a nonzero angle. Our two main examples are kaleidoscopic tilings and their dual tilings. Let $\mathcal{R}$ be ring which will be one of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, or a finite ring $\mathbb{Z}_n$ or finite field $\mathbb{F}_q$. Give all of the tiles on the surface a consistent orientation. Construct the following $\mathcal{R}$-modules:

$$\begin{aligned} C_2(S;\mathcal{R}) &= \mathcal{R}-\text{ linear combinations of oriented tiles,} \\ C_1(S;\mathcal{R}) &= \mathcal{R}-\text{ linear combinations of oriented edges,} \\ C_0(S;\mathcal{R}) &= \mathcal{R}-\text{ linear combinations of vertices, and} \\ C_n(S;\mathcal{R}) &= 0 \text{ otherwise.} \end{aligned}$$

If Δ and e are a tile and edge respectively then $-\Delta$ and $-e$ represent the oppositely oriented objects respectively. An element of $C_n(S;\mathcal{R})$ is called a n-chain. The 1-chains and 2-chains represent various objects on the surface. For example let $e_1, e_2, \ldots, e_k$ be the sequence of edges encountered along a path from A to B on the surface, say $e_i = \overrightarrow{P_i P_{i+1}}$, $P_1 = A$, $P_{k+1} = B$.. Each of the edges has an orientation consistent with direction of travel along the path. Then the path may be represented as the 1-chain $e_1 + e_2 + \cdots + e_k$. The same path with opposite orientation is $-(e_1 + e_2 + \cdots + e_k)$. For an example of a 2-chain let $\Delta_1, \Delta_2, \ldots, \Delta_k$ be a sequence of coherently oriented triangles (e.g., clockwise-oriented triangles on a sphere) such that $\cup_j \Delta_j$ is a region on the surface. Then the region, with appropriate orientation is represented by the 2-chain $\Delta_1 + \Delta_2 + \cdots + \Delta_k$. To give the region the opposite orientation take $-(\Delta_1 + \Delta_2 + \cdots + \Delta_k)$.

Define a sequence of boundary operators:

$$\partial_n : C_n(S;\mathcal{R}) \to C_{n-1}(S;\mathcal{R}).$$

as follows:
Let $\Delta = \Delta(P,Q,R)$ be a positively oriented triangle whose oriented edges are $p = \overrightarrow{QR},\ q = \overrightarrow{RP}, r = \overrightarrow{PQ}$ as we move around the triangle in the positively oriented direction as pictured in Figure 7.1. Then

$$\begin{aligned}
\partial_2(\Delta) &= 1\cdot p + 1\cdot q + 1\cdot r,\\
\partial_1(p) &= 1\cdot R - 1\cdot Q,\\
\partial_1(q) &= 1\cdot P - 1\cdot R,\\
\partial_1(r) &= 1\cdot Q - 1\cdot P.
\end{aligned}$$

The 1 and -1 coefficients are from $\mathcal{R}$; so ,in particular, if $\mathcal{R}$ =$\mathbb{Z}_2$ then the -1's are replaced by 1's. If Δ is a polygon and the edges are $e_1, \ldots e_n$ are we move around the boundary of the polygon in the positively oriented direction then the boundary operator is defined by:

$$\partial_2(\Delta) = 1\cdot e_1 + \cdots + 1\cdot e_n.$$

These operators are extended linearly to the various $C_n(S;\mathcal{R})$ and we usually abbreviate ∂_n to ∂. For example if $e_1 + e_2 + \cdots + e_k$ represents a path as discussed above then

$$\begin{aligned}
\partial(e_1 + e_2 + \cdots + e_k) &= \partial e_1 + \partial e_2 + \cdots + \partial e_k\\
&= P_2 - P_1 + P_3 - P_2 + \cdots + P_{k+1} - P_k\\
&= P_{k+1} - P_1 = B - A.
\end{aligned}$$

Similarly, $\partial(\Delta_1 + \Delta_2 + \cdots + \Delta_k)$ is the oriented boundary of the region $\cup_j \Delta_j$. As an example the reader is invited to find the boundary of the union of the two triangles in Figure 7.1.

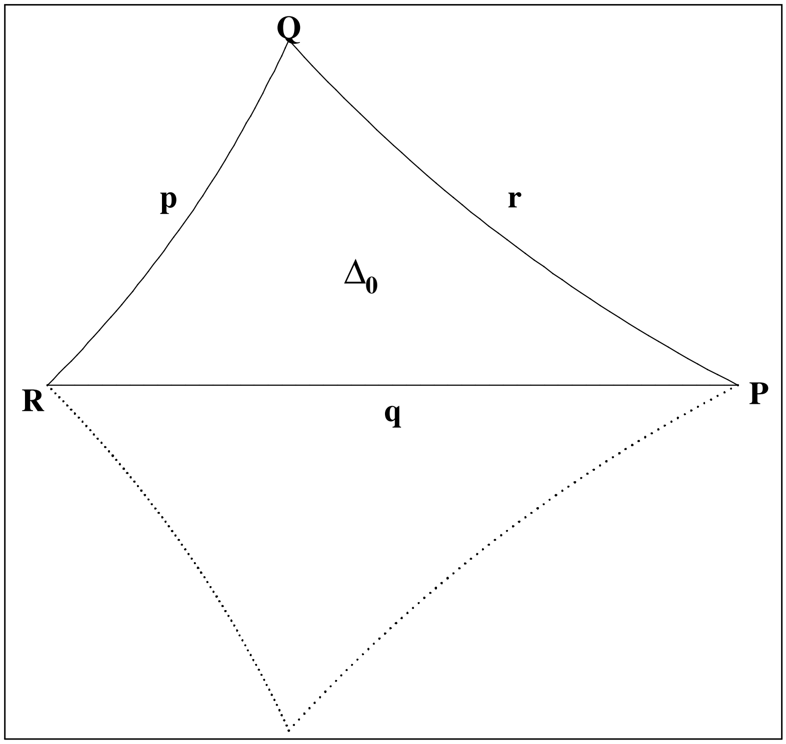

Figure 7.1 A tile, boundary edges and vertices

Using the boundary operator we define two submodules of $C_n(S;\mathcal{R})$.

$$\begin{aligned} Z_n(S;\mathcal{R}) &= \ker \partial_n : C_n(S;\mathcal{R}) \to C_{n-1}(S;\mathcal{R}), \\ B_n(S;\mathcal{R}) &= \mathrm{im}\partial_n : C_{n+1}(S;\mathcal{R}) \to C_n(S;\mathcal{R}). \end{aligned}$$

The elements of $Z_n(S;\mathcal{R})$ are called n-cycles and those of $B_n(S;\mathcal{R})$ are called n-boundaries. For surfaces the objects of interest are 1-cycles and 1-boundaries. From the formula $\partial(e_1 + e_2 + \cdots + e_k) = A - B$ we see that a closed path is a 1-cycle. Likewise a union of closed paths is represented by a cycle. For example the ovals and mirrors on a surface are cycles.

Another example of a cycle is the boundary of a region. In fact it is always true that a 1-boundary is a 1-cycle. More generally $\partial_{n-1}\partial_n = 0$, for all n, or more succinctly $\partial^2 = 0$. This is easily shown for a triangle. In fact, algebraically:

$$\begin{aligned} \partial^2(\Delta) &= \partial(\partial\Delta) = \partial(1 \cdot p + 1 \cdot q + 1 \cdot r) \\ &= 1 \cdot (1 \cdot R - 1 \cdot Q) + 1 \cdot (1 \cdot P - 1 \cdot R) + 1 \cdot (1 \cdot Q - 1 \cdot P) \\ &= R - Q + P - R + Q - P = 0. \end{aligned}$$

Now the result extends by linearity.

We can get an idea about how homology is interpreted by considering the tiling on a torus in Figure 7.2 below. It will also make a clear distinction between cycles and boundaries.

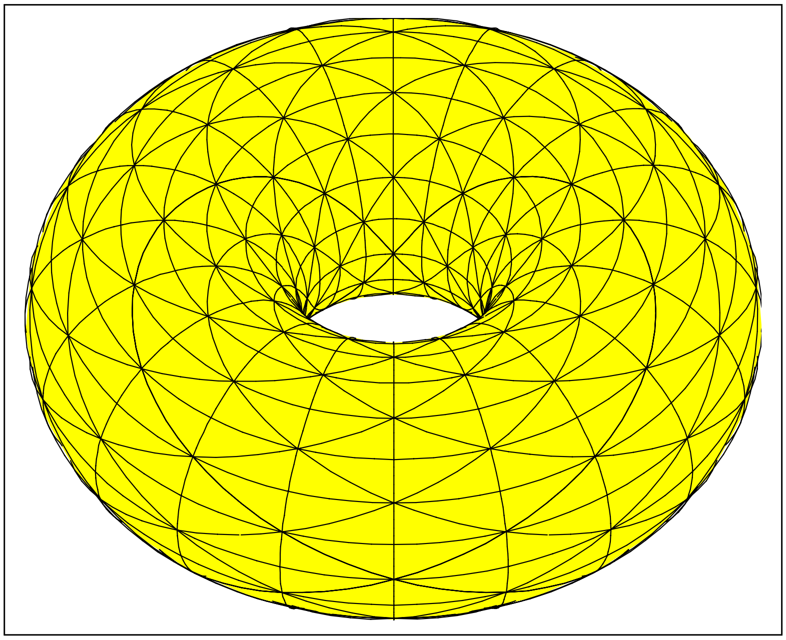

Figure 7.2 Torus with tiling.

Consider the mirror obtained by chopping the torus in half with a vertical plane perpendicular to the page. The intersection is the union of two ovals $\mathcal{O}_1 \cup \mathcal{O}_2$ with $\mathcal{O}_1$ the lower front oval. Give the ovals the orientation induced by travelling up along the inside of the torus out across the top down along the outer portion and then back inwards along the bottom. Since every circle is clearly a cycle, both ovals and the mirror are cycles. Now take the left half of the torus T_1 and give each triangle a clockwise orientation, do the same for the right half of the torus T_2. Let $T_1, T_2, \mathcal{O}_1$, and $\mathcal{O}_2$ also represent the corresponding 2-chains and 1-chains. Then we get

$$\partial T_1 = \mathcal{O}_1 - \mathcal{O}_2,\ \partial T_2 = \mathcal{O}_2 - \mathcal{O}_1,\ \partial \mathcal{O}_1 = 0,\ \partial \mathcal{O}_2 = 0.$$

Now note two relations:

$$\partial(T_1 + T_2) = 0,\ \mathcal{O}_2 = \mathcal{O}_1 + \partial T_1,$$

The first says that the sum of all the triangles with a consistent orientation is a 2-cycle. This is essentially the only 2-cycle. The second says that $\mathcal{O}_1$ and $\mathcal{O}_2$ are homologous - think of $\mathcal{O}_1$ as an elastic band around a torus that we slide around to the position $\mathcal{O}_2$. In the 1-form version of Stokes theorem this has the following interpretation:

$$\int\limits_{\mathcal{O}_2} \omega = \int\limits_{\mathcal{O}_1 + \partial T_1} \omega = \int\limits_{\mathcal{O}_1} \omega + \int\limits_{\partial T_1} \omega = \int\limits_{\mathcal{O}_1} \omega + \iint\limits_{T_2} d\omega.$$

If ω is a closed form, i.e., $d\omega = 0$ then

$$\int\limits_{\mathcal{O}_2} \omega = \int\limits_{\mathcal{O}_1} \omega.$$

From this point of view we wish to identify homologous cycles. The homology modules do just that by taking quotient modules.

Definition 7.1 Let S be a surface with a tiling and let other notions be as above. The homology groups of with coefficients in $\mathcal{R}$ are defined by:

$$H_n(S;\mathcal{R}) = \frac{Z_n(S;\mathcal{R})}{B_n(S;\mathcal{R})}.$$

Here is the main theorem on the homology groups of surfaces.

Proposition 7.1 *Let S be a connected, closed, orientable surface of genus σ. Then,*

$$\begin{aligned} H_0(S;\mathcal{R}) &\cong \mathcal{R}, \\ H_1(S;\mathcal{R}) &\cong \mathcal{R}^{2\sigma}, \\ H_2(S;\mathcal{R}) &\cong \mathcal{R}, \end{aligned}$$

and all other $H_n(S;\mathcal{R})$ are 0. In particular the homology groups are independent of the tiling used to compute it.

In the example above we saw that the mirror of the reflection split the torus in two and also that a linear combination of the ovals is a boundary: namely $\mathcal{O}_1 - \mathcal{O}_2 = \partial T_1$. This will be true in general. In fact if we split a surface in half by cutting along a mirror then the boundary of one half will be a linear combination of ovals equal to a mirror. The converse is also true:

Proposition 7.2 *A reflection on a surface is separating if and only if the sum of the ovals (appropriately oriented) in the mirror in homologous to zero (equals* 0 *in $H_1(S;\mathcal{R})$).*

Remark 7.1 Suppose that we have two tilings that have a common refinement. The canonical example of this is the tiling obtained by imposing the kaleidoscopic tiling and the dual tiling on the surface at the same time. This cuts up each triangle into 3 quadrilaterals. The edges and vertices are obtained by taking edges and vertices of these quadrilaterals. Each tile of the kaleidoscopic or the dual tiling is a union of tiles of the refinement. The same holds true of edges. This allows us to simultaneously consider constructions in both tilings.

7.2 Binary coefficients and the intersection form

If we select $\mathcal{R} = \mathbb{F}_2$ then the issues of orientation noted above may be ignored and we are then really only working with subsets of the totality of tiles, edges and vertices. Often this will be all we need. With binary coefficients we can easily define a bilinear form on $H_1(S;\mathcal{R})$. The bilinear form exists for every ring $\mathcal{R}$ but it is easier to define for binary coefficients.

Definition 7.2 Suppose that $x, y \in H_1(S; \mathcal{R})$. Let x' y' denote any two cycles representing the respective homology classes. Then, we define the intersection product of $x \cdot y$ to be:

$$x \cdot y = \# x' \cap y' \bmod 2.$$

Remark 7.2 The intersection product detects whether two closed curves meet an even or odd number of times. The product $x \cdot y$ doesn't depend on the homology classes chosen. This requires proving that every cycle meets a boundary an even number of times.

Proposition 7.3 *The intersection product for a general ring $\mathcal{R}$ is a non-degenerate skew-symmetric, bilinear form. This means:*

$$\begin{aligned} x \cdot y &= -y \cdot x \\ (ax + by) \cdot z &= ax \cdot z + by \cdot z, \ a, b \in \mathcal{R}, \\ \textit{if } x \cdot y &= 0 \textit{ for all } x \in H_1(S; \mathcal{R}) \textit{ then } y = 0. \end{aligned}$$

For binary coefficients the first two become:

$$\begin{aligned} x \cdot y &= y \cdot x, \\ (x + y) \cdot z &= x \cdot z + y \cdot z. \end{aligned}$$

The non-degeneracy condition becomes: a cycle is a boundary if and only if it meets every other cycle in an even number of points.

The criterion for separability of a reflection given above may be rephrased in terms of the intersection product (direct geometric proofs also exist).

Proposition 7.4 *A reflection is separating if and only if the sum of its ovals is homologous to* 0 mod 2. *Stated otherwise, a reflection is separating if and only if every closed path cross the mirror of the symmetry an even number of times.*

Remark 7.3 By considering a common refinement of the kaleidoscopic and the dual tiling as described in Remark**??** it is possible to define and compute the intersection product of a kaleidoscopic cycle with a dual cycle. These intersection products are easy to calculate since a kaleidoscopic edge e and a dual edge f are either disjoint of intersect at right angles in their interiors. Moreover for each kaleidoscopic edge e there is a unique dual edge f such that $e \cdot f = 1$ and vice versa.

7.3 The group action on homology

Each $g \in G^*$ permutes the edges of either the kaleidoscopic or the dual tiling and hence a defines a linear transformation of the 1-chain module by

$$a_1 e_1 + a_2 e_2 + \cdots + a_k e_k \to a_1 g e_1 + a_2 g e_2 + \cdots + a_k g e_k.$$

Similar formulas apply for tiles and vertices. It is pretty easy to see that the G^*-action commutes with the boundary operator "the transform of the boundary is the boundary of the transform" or $\partial gc = g\partial c$ for every chain. This means then that $gZ_n(S;\mathcal{R}) = Z_n(S;\mathcal{R})$ and $gB_n(S;\mathcal{R}) = B_n(S;\mathcal{R})$. As a consequence it follows that each element of G^* maps homology classes to homology classes. This gives rise to a linear action of G^* on $H_n(S;\mathcal{R})$, .i.e.,

$$\begin{aligned} g\zeta &\in H_n(S;\mathcal{R}) \text{ for } \zeta \in H_n(S;\mathcal{R}), \\ g(h\zeta) &= (gh)\zeta, \text{ for } g,h \in G^*, \\ g(a\zeta + b\xi) &= ag\zeta + bg\xi, \text{ for } g \in G^*, \ \zeta,\xi \in H_n(S;\mathcal{R}), .a,b \in \mathcal{R}. \end{aligned}$$

A submodule of $V \subseteq H_n(S;\mathcal{R})$ is said to be G^* -invariant or a G^*-submodule if $gV = V$ for all $g \in G^*$. This action of G^* on the homology is called the *homology representation.* See Remark 7.12 and Chapter 8 and for more on representations, induced by G and G^*.

Remark 7.4 If $\mathcal{R}$ is a field then we are working with vector spaces in which many things are simpler. If $\mathcal{R}$ is a field of characteristic 0 then things are under good control. Unfortunately for $\mathcal{R} = \mathbb{F}_2$ the simplification we get for studying the surface destroys the clean theory of representation over fields of characteristic 0.

Remark 7.5 Though we need not go into it here, the homology representation of G can be easily calculated from the generating triples (see [3]).

Example 7.1 Let S be the surface of genus 3 with Hurwitz tiling $(2,3,7)$. Then $G = PSL_2(7)$ and $H_1(S;\mathbb{C})$ is the direct sum of two invariant G-submodules.

Example 7.2 Let $\mathcal{O}_1,\ldots,\mathcal{O}_K$ be the collection of ovals of reflections. Each oval determines a homology class in $H_1(S;\mathcal{R})$. Let $\mathcal{O}(S;\mathcal{R})$ denote the linear span of these homology classes Then $\mathcal{O}(S;\mathcal{R})$ is a G^*-submodule of $H_1(S;\mathcal{R})$ called the *oval submodule.*

Example 7.3 We can construct a variant $\mathcal{O}(S;\mathcal{R})$ by taking only the linear span of ovals coming from reflections conjugate to a given reflection.

Example 7.4 If V is a G^*-submodule of $H_1(S;\mathcal{R})$ then $V^\perp = \{x \in H_1(S;\mathcal{R}) : x \cdot y = 0 \text{ for all } y \in V\}$ is a G^*-submodule of $H_1(S;\mathcal{R})$.

Example 7.5 Let $\mathcal{M} = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_s$ be the mirror of a reflection r_e. If $\mathcal{R} = \mathbb{F}_2$ then $\mathcal{O}_1 + \cdots + \mathcal{O}_s$ is the only possible linear combination of all of the ovals of $\mathcal{M}$. Now take the linear span of all these mirrors as homology classes to create the *mirror submodule.*We may also form variants of this submodule by taking the linear span of mirrors over a conjugacy class of reflections.

7.4 The Hurewicz homomorphism and covers

Recall that $S = U/\Gamma$ where $\Gamma \lhd \Lambda^*$ is the fundamental group and U is the universal cover. Previously we associated a path of edges from the dual tiling on S based at I_0 with an element $g \in \Gamma$. This path is a cycle and hence determines a integral homology class on S, i.e., an element of $H_1(S;\mathbb{Z})$. It is not to hard to show that the element g determines a unique homology class $\overline{g}$ in $H_1(S;\mathbb{Z})$ and that the map $\mathcal{H} : \Gamma \to H_1(S;\mathbb{Z})$ is a surjective group homomorphism.

Proposition 7.5 *Let $\mathcal{H} : \Gamma \to H_1(S;\mathbb{Z})$ be the Hurewicz map above. Then:*

- $\mathcal{H}$ is surjective and $\ker \mathcal{H} = [\Gamma, \Gamma]$.
- Every map $\Gamma \to A$ where A is an abelian group factors through homology: $\Gamma \to H_1(S;\mathbb{Z}) \to A$.

Let $\rho_n : H_1(S;\mathbb{Z}) \to H_1(S;\mathbb{Z}_n)$ be the map obtained by reducing coefficients mod n. Then, there is a map

$$\rho_n : \Gamma \to H_1(S;\mathbb{Z}) \to H_1(S;\mathbb{Z}_n).$$

The latter is a finite abelian group. It is not to hard to prove that for $g \in \Lambda^*$, $h \in \Gamma$ that:

$$\rho_n(ghg^{-1}) = \eta(g)\rho_n(h),$$

where $\eta : \Lambda^* \to G^* = \Lambda^*/\Gamma$ is the quotient map. Using this equation we obtain a collection of covers called homology covers:

Proposition 7.6 *Let $V \subseteq H_1(S;\mathbb{Z}_n)$ be a submodule. The subgroup $\widetilde{\Gamma} = \rho_n^{-1}(V)$ is normal in Λ^* if and only if V is G^*-submodule of $H_1(S;\mathbb{Z}_n)$. Thus for each G^*-submodule V of $H_1(S;\mathbb{Z}_n)$ there is Galois cover $\widetilde{S} = U/\widetilde{\Gamma} \to U/\Gamma = S$ such that the tiling lifted to $\widetilde{S}$ is kaleidoscopic and such that the Galois group of covering $\widetilde{S} \to S$ is*

$$N = \Gamma/\widetilde{\Gamma} = H_1(S;\mathbb{Z}_n)/V. \tag{1}$$

In terms of the tiling groups $\widetilde{G}^*$ and G^* we get an exact sequence:

$$N \to \widetilde{G}^* \to G^* \tag{2}$$

Remark 7.6 If V is just a submodule of $H_1(S;\mathbb{Z}_n)$ but not a G^*-submodule then $\widetilde{\Gamma} = \rho_n^{-1}(V)$ is normal in Γ but not in Λ^*. The covering $\widetilde{S} \to S$ exists, it is still a Galois cover and 7.1 still holds. However the tiling group $\widetilde{G}^*$ does not exist.

Separability and the Hurewicz map The following was introduced in **??** in an attempt to analyze separability. Let $M = S_\phi$ be the mirror of the reflection ϕ, and let M also denote the homology class in $H_1(S;\mathbb{Z}_2) = H_1(S;\mathbb{F}_2)$ by taking the sum of all the edges. If $\gamma = f_1 f_2 \cdots f_n$ is any path then $\rho_2(\gamma) = f_1 + f_2 + \cdots + f_n$ as a 1-cycle. The number of times that γ crosses the mirror mod 2 is given by: $\iota_M(\gamma) = \rho_2(\gamma) \cdot M = (f_1 + f_2 + \cdots + f_n) \cdot M$. We get the following.

Proposition 7.7 *Let Γ_ϕ be the subset of Γ consisting of elements which cross the mirror M and even number of times. Then the sequence is exact.*

$$\Gamma_\phi \to \Gamma \xrightarrow{\iota_M} \mathbb{F}_2$$

is exact. The mirror M splits S if and only if $\Gamma_\phi = \Gamma$. Otherwise, $\Gamma/\Gamma_\phi = \mathbb{F}_2$.

The following is proved in **??**.

Proposition 7.8 *Let M^ϕ be the inverse image of M under the cover $S^\phi \to S$ defined by the inclusion $\Gamma_\phi \subseteq \Gamma$. Then M^ϕ separates S^ϕ.*

As noted in **??** the lift of the tiling to S^ϕ need not be kaleidoscopic if Γ_ϕ is not normal in Λ. This may be fixed up as follows.

Proposition 7.9 *Let $\mathcal{M}$ denote the mirror submodule, constructed from the mirrors of a conjugacy class of reflections, and let $\mathcal{M}^\perp$ be its orthogonal complement. Let $\widetilde{\Gamma} = \rho_n^{-1}(\mathcal{M}^\perp)$. Then the associated cover $\widetilde{S} \to S$ is such that the lift $\widetilde{M}$ splits $\widetilde{S}$ for every mirror M in the conjugacy class, and the full tiling group $\widetilde{G}^*$ on $\widetilde{S}$ satisfies equations 7.1 and 7.2.*

7.5 Explicit homology calculation for the dual tiling

In this section we show how to compute Γ/Γ_ϕ and $\Gamma/\widetilde{\Gamma}$ in the two previous propositions. Now observe that since every cycle is orthogonal to the boundaries then $B_n(S;\mathbb{F}_2) \perp M$ for every mirror M. Therefore

$$\begin{aligned}
\Gamma/\Gamma_\phi &\cong \frac{Z_1(S;\mathbb{F}_2)}{Z_1(S;\mathbb{F}_2)\cap M^\perp}, \text{ and} \\
\Gamma/\widetilde{\Gamma} &\cong \frac{Z_1(S;\mathbb{F}_2)}{Z_1(S;\mathbb{F}_2)\cap \mathcal{M}^\perp}.
\end{aligned}$$

We will identify $Z_n(S;\mathcal{R})$ and $M^\perp$ as the null-space of certain matrices. Then the dimension of Γ/Γ_ϕ and $\Gamma/\widetilde{\Gamma}$ as $\mathbb{F}_2$ can be identified by taking the difference of the ranks of these matrices. First a little setup.

Let f_p, f_q, and f_r denote the edges in the dual tiling that cross the edges p, q, and r of the master tile, respectively. Denote these edges of the master tile by e_p, e_q, and e_r, respectively. Now observe that the edges of the dual tiling and the kaleidoscopic tiling are each unions of three orbits (defined by types):

$$\begin{aligned}
\{\text{dual edges}\} &= Gf_p \cup Gf_q \cup Gf_r \text{ and} \\
\{\text{kaleidoscopic edges}\} &= Ge_p \cup Ge_q \cup Ge_r.
\end{aligned}$$

Furthermore, no edge is fixed by a non-trivial element of G. The vertices of the dual tiling are the single orbit $G^* I_0$ where I_0 is the incenter of the master tile. No vertex is fixed by any element of G^*. Thus an element of $\alpha \in C_1(S;\mathbb{F}_2)$ and $\beta \in C_0(S;\mathbb{F}_2)$ have the form

$$\begin{aligned}
\alpha &= \sum_{g\in G}(x_g g f_p + y_g g f_q + z_g g f_r), \text{ and} \\
\beta &= \sum_{g\in G}(u_g g I_0 + v_g g p I_0).
\end{aligned}$$

Now a simple calculation shows that

$$\begin{aligned}
\partial(g f_p) &= g\partial(f_p) = gI_0 + gpI_0, \\
\partial(g f_q) &= g\partial(f_q) = gI_0 + gqI_0 = gI_0 + ga^{-1}pI_0, \\
\partial(g f_r) &= g\partial(f_r) = gI_0 + grI_0 = gI_0 + gcpI_0,
\end{aligned}$$

If α is a cycle then

$$\begin{aligned}
0 &= \partial\alpha \\
&= \sum_{g\in G}(x_g\partial(gf_p) + y_g\partial(gf_q) + z_g\partial(gf_r)) \\
&= \sum_{g\in G}(x_g(gI_0 + gpI_0) + y_g(gI_0 + ga^{-1}pI_0) + z_g(gI_0 + gcpI_0)) \\
&= \sum_{g\in G}(x_g + y_g + z_g)gI_0 + \sum_{g\in G} x_g gpI_0 + \sum_{g\in G} y_g ga^{-1}pI_0 + \sum_{g\in G} z_g gcpI_0
\end{aligned}$$

Replace g by ha and kc^{-1} in the third and fourth sums respectively. We obtain:

$$\begin{aligned}
0 &= \sum_{g\in G}(x_g + y_g + z_g)gI_0 + \sum_{g\in G} x_g gpI_0 + \sum_{ha\in G} y_{ha} hpI_0 + \sum_{kc^{-1}\in G} z_{kc^{-1}} kcpI_0. \\
&= \sum_{g\in G}(x_g + y_g + z_g)gI_0 + \sum_{g\in G} x_g gpI_0 + \sum_{g\in G} y_{ga} gpI_0 + \sum_{g\in G} z_{kc^{-1}} gcpI_0. \\
&= \sum_{g\in G}(x_g + y_g + z_g)gI_0 + \sum_{g\in G}(x_g + y_{ga} + z_{kc^{-1}})gcpI_0.
\end{aligned}$$

The middle line is obtained by reordering the summations and choosing a new summation variable. The points gI_0 and $gcpI_0$ are all distinct, indeed the collection $\{hI_0 : h \in G^*\}$ is a basis for $C_1(S,\mathbb{F}_2)$. Therefore we get the following sets of linear equations that hold if and only if α is a cycle.

$$x_g + y_g + z_g = 0, \; g \in G, \tag{7.3}$$
$$x_g + y_{ga} + z_{kc^{-1}} = 0, \; g \in G. \tag{7.4}$$

Remark 7.7 The equations have the following interpretation. Color the tiles white or black according to whether the tile is in the orbit $G\Delta_0$ or $Gp\Delta_0$. Label each edge of the dual tiling with x_g, y_g, or z_g according to the coefficient in the expansion of α. The first equation says that the sum of the labels on edges at the vertex in a white tile is zero. The reader is invited to show that the second set of equation says that the sum of the edge labels that at the vertex in a black tile is zero. Alternatively, think of the edges as a system of wires and the that the coefficients measure the flow of electrical current in the wires. The cycle equations say that there is no net accumulation at any node which is just Kirchoff's current law. For this to make sense we should pick $\mathcal{R} = \mathbb{R}$ and we have to use the boundary equations below, thinking of gf_p as an edge from gpI_0 to gI_0, and similar conditions on the edges gf_q and gf_r.

$$\begin{aligned}
\partial(gf_p) &= g\partial(f_p) = gI_0 - gpI_0, \\
\partial(gf_q) &= g\partial(f_q) = gI_0 - ga^{-1}pI_0, \\
\partial(gf_r) &= g\partial(f_r) = gI_0 - gcpI_0.
\end{aligned}$$

Now let us consider the calculations required to compute $M^{\perp}$. We represent M as a cycle by

$$\mu = \sum_{g \in G} (a_g g e_p + b_g g e_q + c_g g e_r).$$

As $ge_p \cdot gf_p = ge_q \cdot gf_q = ge_r \cdot gf_r = 1$ and all other $he_s \cdot kf_t = 0$ for $h, k \in G$ and $s, t \in \{p, q, r\}$, then

$$\mu \cdot \alpha = \sum_{g \in G} (a_g x_g + b_g y_g + c_g z_g). \tag{5}$$

Now let us pick an ordering of the elements of $G = \{g_1, g_2, \ldots, g_{|G|}\}$,and form a vector of coefficients $\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ as follows:

$$X = \begin{bmatrix} x_{g_1} \\ x_{g_2} \\ \vdots \\ x_{g_{|G|}} \end{bmatrix}, \ Y = \begin{bmatrix} y_{g_1} \\ y_{g_2} \\ \vdots \\ y_{g_{|G|}} \end{bmatrix}, \ Z = \begin{bmatrix} z_{g_1} \\ z_{g_2} \\ \vdots \\ z_{g_{|G|}} \end{bmatrix}.$$

If we order the cycle equations 7.3 and 7.4 according to the ordering of our group then the cycle equations have the form:

$$\begin{bmatrix} I & I & I \\ I & I^a & I^{c^{-1}} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

where I is the $|G| \times |G|$ identity matrix and I^h the matrix obtained by permuting the columns of I according to the permutation defined by $g_{i'} = g_i h$. We have the following Proposition.

Proposition 7.10 *Let notation be as immediately above. Then the cycle space $H_1(S; \mathbb{F}_2)$ may be identified with the null space of the matrix*

$$\begin{bmatrix} I & I & I \\ I & I^a & I^{c^{-1}} \end{bmatrix}.$$

Next let

$$\begin{aligned} A_M &= \left[\begin{array}{cccc} a_{g_1} & a_{g_2} & \cdots & a_{g_{|G|}} \end{array} \right], \\ B_M &= \left[\begin{array}{cccc} b_{g_1} & b_{g_2} & \cdots & b_{g_{|G|}} \end{array} \right], \\ B_M &= \left[\begin{array}{cccc} c_{g_1} & c_{g_2} & \cdots & a_{g_{|G|}} \end{array} \right]. \end{aligned}$$

Then equation 7.5 becomes

$$\left[\begin{array}{ccc} A_M & B_M & B_M \end{array} \right] \left[\begin{array}{c} X \\ Y \\ Z \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right].$$

We get the following proposition:

Proposition 7.11 *Let notation be as immediately above. Then the orthogonal complement of the mirror cycle $M^\perp$ may be identified with the null space of the matrix*

$$\left[\begin{array}{ccc} A_M & B_M & C_M \end{array} \right].$$

Now we know that $r_{ge} = gr_e g^{-1}$. Therefore, if follows that if $\mu = \sum_{h \in G}(a_h h e_p + b_h h e_q + c_h h e_r)$ is the cycle representing the mirror of r_e then

$$\begin{aligned} g\mu &= \sum_{h \in G}(a_h g h e_p + b_h g h e_q + c_h g h e_r) \\ &= \sum_{h \in G}(a_{g^{-1}h} h e_p + b_{g^{-1}h} h e_q + c_{g^{-1}h} h e_r) \end{aligned}$$

if the cycle representing $gr_e g^{-1}$. Slightly less cumbersome to show is that $g^{-1}\mu = \sum_{h \in G}(a_{gh} h e_p + b_{gh} h e_q + c_{gh} h e_r)$ is the cycle corresponding to the mirror of $g^{-1} r_e g$. Thus the orthogonal complement of the mirror submodule $\mathcal{M}^\perp$ is the solution to the equations:

$$\left[\begin{array}{ccc} A_M & B_M & C_M \\ {}^{g_1}A_M & {}^{g_1}B_M & {}^{g_1}C_M \\ \vdots & \vdots & \vdots \\ {}^{g_{|G|}}A_M & {}^{g_{|G|}}B_M & {}^{g_{|G|}}C_M \end{array} \right] \left[\begin{array}{c} X \\ Y \\ Z \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$$

where each row of the left matrix is permuted according to the left action on $G : x \to gx$. We have another proposition:

Proposition 7.12 *Let notation be as immediately above. Then the orthogonal complement of the mirror submodule $\mathcal{M}^\perp$ for a conjugacy class of mirrors may be identified with the null space of the matrix*

$$\left[\begin{array}{ccc} A_M & B_M & C_M \\ {}^{g_1}A_M & {}^{g_1}B_M & {}^{g_1}C_M \\ \vdots & \vdots & \vdots \\ {}^{g_{|G|}}A_M & {}^{g_{|G|}}B_M & {}^{g_{|G|}}C_M \end{array} \right].$$

Example 7.6 Let $G = \mathbb{Z}_5$ with generator x. For a generating vector we pick $(a, b, c) = (x, x, x^3)$. The automorphism θ is $\theta(g) = g^{-1}$ as we must always have for an abelian group. All reflections are conjugate since l, m and n are odd. The cycle corresponding to the mirror of p is

$$\begin{aligned} \mu_p &= e_p + a^3 e_q + a^3 b^3 e_r \\ &= e_p + x^3 e_q + x e_r. \end{aligned}$$

Let us order the elements of G by $g_i = x^i$. Then the cycle matrix is

$$C = \begin{bmatrix} 1&0&0&0&0&1&0&0&0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&1&0&0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&0&1&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0&0&1&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0&0&0&1&0&0&0&0&1 \\ 1&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&1&0&0&0&0&1&0&0 \\ 0&0&1&0&0&0&0&0&1&0&0&0&0&1&0 \\ 0&0&0&1&0&0&0&0&0&1&1&0&0&0&1 \\ 0&0&0&0&1&1&0&0&0&0&0&1&0&0&0 \end{bmatrix}$$

The matrix for determining the orthogonal complement of the oval is:

$$B = \begin{bmatrix} 1&0&0&0&0&0&0&1&0&0&0&1&0&0&0 \end{bmatrix}$$

and the matrix for determining the orthogonal complement of the of the mirror submodule is

$$B_5 = \begin{bmatrix} 1&0&0&0&0&0&0&0&1&0&0&1&0&0&0 \\ 0&1&0&0&0&0&0&0&0&1&0&0&1&0&0 \\ 0&0&1&0&0&1&0&0&0&0&0&0&0&1&0 \\ 0&0&0&1&0&0&1&0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&1&0&0&1&0&0&0&0 \end{bmatrix}.$$

Now we have:

$$\begin{aligned} \dim(Z_1(S; \mathbb{F}_2)) &= 15 - \operatorname{rank}(C), \\ \dim(Z_1(S; \mathbb{F}_2) \cap M^{\perp}) &= 15 - \operatorname{rank}\left(\begin{bmatrix} C \\ B \end{bmatrix} \right), \\ \dim(Z_1(S; \mathbb{F}_2) \cap \mathcal{M}^{\perp}) &= 15 - \operatorname{rank}\left(\begin{bmatrix} C \\ B_5 \end{bmatrix} \right). \end{aligned}$$

By Maple calculation the echelon form of C, $\begin{bmatrix} C \\ B \end{bmatrix}$, and $\begin{bmatrix} C \\ B_5 \end{bmatrix}$ are given by:

$$C \to \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$\begin{bmatrix} C \\ B \end{bmatrix} \longrightarrow \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$\begin{bmatrix} C \\ B_5 \end{bmatrix} \to \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

Therefore the ranks are given by:

$$\begin{aligned}
\mathrm{rank}(C) &= 9, \\
\mathrm{rank}\left(\begin{bmatrix} C \\ B \end{bmatrix}\right) &= 10, \\
\mathrm{rank}\left(\begin{bmatrix} C \\ B_5 \end{bmatrix}\right) &= 11.
\end{aligned}$$

Thus

$$\begin{aligned}
\Gamma/\Gamma_\phi &\cong \mathbb{F}_2 \\
\Gamma/\widetilde{\Gamma} &\cong \mathbb{F}_2^2
\end{aligned}$$

7.6 Cohomology and conformal group actions

The cohomology of a surface is the linear dual to its homology. This may be constructed in two ways, the first as differential forms and the second from the linear dual of homology of the tiling. The first allows the powerful ideas of differential geometry and analysis to be used. The second allows us to have arbitrary coefficients and to use the geometry of the tilings.

Differential forms First we examine the differential form definition. There are three types of differential forms on a surface. In local coordinates x, y on the surface, the 0-forms are differentiable functions $f = f(x, y)$, 1-forms have the local representation $gdx + hdy = g(x, y)dx + h(x, y)dy$ and 2-forms have the local representation: $kdx \wedge dy = k(x, y)dx \wedge dy$. In local coordinates the exterior derivative of these forms is given by:

$$\begin{aligned}
df &= f_x dx + f_y dy \\
d(gdx + hdy) &= dg \wedge dx + dh \wedge dy \\
&= (g_x dx + g_y dy) \wedge dx + (h_x dx + h_y dy) \wedge dy \\
&= g_x dx \wedge dx + g_y dy \wedge dx + h_x dx \wedge dy + h_y dx \wedge dy \\
&= (h_x - g_y) dx \wedge dy \\
d(kdx \wedge dy) &= 0.
\end{aligned}$$

The general Stokes theorem says

$$\int\limits_W d\omega = \int\limits_{\partial W} \omega$$

where ω is an n-form, W is a $(n+1)$-dimensional oriented region or submanifold with boundary. The integral symbol denotes an $n+1$ and n-dimensional integration on the left and right hand sides respectively. Specifically for $n = 0, 1$ we get two instances of Stokes theorem. If W is the oriented path x_0 to x_1 then:

$$\int_W df = f(P_1) - f(P_0),$$

and for a region W with boundary ∂W

$$\iint\limits_{W} d\omega = \int\limits_{\partial W} \omega.$$

To define cohomology as a dual to homology we need to dualize the ideas of boundary, cycle, homologous, and homology group. Clearly, from Stokes theorem, the differential is a good candidate for the dual notion of the boundary operator. Cycles were objects with zero boundary, we now call n-cocycle a differential n-form ω satisfying $d\omega = 0$, i.e., a closed form. Note that the 0-cocylces are constant functions and that every 2-form is automatically closed. Analogously, the form $d\omega$ is called a $(n+1)$-coboundary, or an exact form. The only 0-coboundary is the zero function on S. Now define the following, for $\mathcal{R} = \mathbb{R}$ or $\mathbb{C}$:

$$\begin{aligned}
\Omega^n(S;\mathcal{R}) &= \mathcal{R}\text{-valued differential } n\text{-forms on } S. \\
Z^n(S;\mathcal{R}) &= \ker(d : \Omega^n(S) \to \Omega^{n+1}(S)) \;=\; \{\text{closed } n\text{-forms}\} \text{ or } \{\text{cocyles}\} \\
B^n(S;\mathcal{R}) &= \operatorname{im}(d : \Omega^{n-1}(S) \to \Omega^n(S)) \;=\; \{\text{exact } n\text{-forms}\} \text{ or } \{\text{coboundaries}\} \\
H^n(S;\mathcal{R}) &= Z^n(S;\mathcal{R})/B^n(S;\mathcal{R}).
\end{aligned}$$

We have an analogue to Proposition 7.1

Proposition 7.13 *Let S be a connected, closed, orientable surface of genus σ, and $\mathcal{R} = \mathbb{R}$ or $\mathbb{C}$. Then:*

$$\begin{aligned}
H^0(S;\mathcal{R}) &\cong \mathcal{R}, \\
H^1(S;\mathcal{R}) &\cong \mathcal{R}^{2\sigma}, \\
H^2(S;\mathcal{R}) &\cong \mathcal{R},
\end{aligned}$$

and all other $H^n(S;\mathcal{R})$ are 0.

Now let us make the duality of the homology and cohomology groups more explicit. Suppose that ω is an n-form and W is an n-chain. Then the integration of ω over W, $\int_W \omega$, is bilinear in the sense that

$$\begin{aligned}
\int_W (a_1\omega_1 + a_2\omega_2) &= a_1 \int_W \omega_1 + a_2 \int_W \omega_2, \\
\int_{b_1W_1+b_2W_2} \omega &= b_2 \int_{W_2} \omega + b_2 \int_{w_2} \omega,
\end{aligned}$$

If ω is closed, so that $d\omega = 0$, and W_1 and W_2 are homologous, i.e. $W_2 = W_1 + \partial V$. Then

$$\int_{W_2} \omega = \int_{W_1} \omega + \int_{\partial V} \omega = \int_{W_1} \omega + \int_V d\omega = \int_{W_1} \omega. \tag{6}$$

Thus $\int_W \omega$ depends only on the homology class of W. Now assume that W is a cycle, so that $\partial W = 0$, and that ω_1 and ω_2 are cohomology forms, i.e., $\omega_2 = \omega_1 + d\psi$.

$$\int_W \omega_2 = \int_W \omega_1 + \int_W d\psi = \int_W \omega_1 + \int_{\partial W} \psi = \int_W \omega_1. \tag{7}$$

Then $\int_W \omega$ depends only on the cohomology class of ω. The observations allow us to define a linear function $\int_W \omega$ on $H_n(S;\mathcal{R})$. To see that this is well defined let $W_2 = W_1 + \partial V$ and $\omega_2 = \omega_1 + d\psi$ be homologous cycles and cohomologous closed forms respectively. Then

$$\int_{W_2} \omega_2 = \int_{W_1} \omega_2 = \int_{W_1} \omega_1$$

by 7.6 and 7.7 respectively. This discussion may be summarized in the following proposition.

Proposition 7.14 *Let S be a connected, closed, orientable surface of genus σ, and $\mathcal{R} = \mathbb{R}$ or $\mathbb{C}$. Then, the linear map $\int : H^n(S;\mathcal{R}) \to \mathrm{Hom}(H_n(S;\mathcal{R}),\mathcal{R})$, $\omega \to \int_W \omega$ is a linear isomorphism. In particular, the cohomology class of a closed 1-form is uniquely determined by the integrals of ω over a homology basis.*

Remark 7.8 By $\mathrm{Hom}(H_n(S;\mathcal{R}),\mathcal{R})$ we mean the $\mathcal{R}$-vector space of $\mathcal{R}$-linear functionals on $H_n(S;\mathcal{R})$. Each such functional is determined by the values at a homology basis.

Cohomology via tilings Now let us look at a direct geometric construction of the cohomology of S. Here we do not need to assume that $\mathcal{R} = \mathbb{R}$ or $\mathbb{C}$. We will use the Remark 7.8 as a starting point. Let

$$C^n(S;\mathcal{R}) = \mathrm{Hom}(C_n(S;\mathcal{R}),\mathcal{R}),$$

they are the n-cochains on S. We may think of cochain as an assignment of "voltages" or "voltage differences" to the n-chains on S, at least in the case of 0-chains and 1-chains. For example, let us consider $f \in C^1(S;\mathcal{R})$ which assigns to each oriented edge e the "voltage difference" $f(e)$. For an arbitrary chain $c = \sum_e a_e e$ a sum of oriented edges $f(c) = \sum_e a_e f(e)$ is the sum of the voltage differences on the chain. In particular, if $e_1 + e_2 + \cdots + e_k$ is the sum of the oriented edges as we move around a cycle of edges on S then the change in voltage will be $f(e_1) + f(e_2) + \cdots + f(e_k)$. The coboundary operator on $\delta : C^n(S;\mathcal{R}) \to C^{n+1}(S;\mathcal{R})$ is defined by

$$\delta f(c) = f(\partial c).$$

So, if $f \in C^0(S;\mathcal{R})$ is a voltage assignment to all the vertices of a tiling, and the e is the oriented edge from P_0 to P_1 then $\delta f(e) = f(P_1) - f(P_0)$ is the voltage difference along the edge. For $f \in C^1(S;\mathcal{R})$ and Δ a tile with $\partial\Delta = e_1 + e_2 + \cdots + e_k$ a sum of oriented edges, then $\delta f(\Delta) = f(e_1) + f(e_2) + \cdots + f(e_k)$ is the sum of the voltage differences around the boundary. As before we define cocylces and coboundaries

$$\begin{aligned} Z^n(S;\mathcal{R}) &= \ker(\delta : C^n(S) \to C^{n+1}(S)) = \{\text{cocyles}\} \\ B^n(S;\mathcal{R}) &= \mathrm{im}(\delta : C^{n-1}(S) \to C^n(S)) = \{\text{coboundaries}\} \\ H^n(S;\mathcal{R}) &= Z^n(S;\mathcal{R})/B^n(S;\mathcal{R}). \end{aligned}$$

Remark 7.9 We may interpret the terms 1-cocycle and 1-coboudary in terms the Kirchoff Voltage laws, analogously to the cycle condition $\partial c = 0$ homology condition being interpreted as a current law in Remark 7.7 . By definition, The 1-cochain f is a cocycle if and only if $f(c) = 0$ whenever c is a cycle occurring as the boundary of a tile $c = \partial\Delta$. It may be proved that f is a coboundary $f = \delta V$ and only if $f(c) = 0$ for each cycle. This is known as the Kirchoff Voltage Law. In the dual tiling cocycle conditions may be interpreted as the Kirchoff current law.

Next we want to show the analogue of Proposition 7.14. Pick a n-cocycle $f \in Z^n(S;\mathcal{R})$ and n-chain $c_1 \in C_n(S;\mathcal{R})$. If $c_2 = c_1 + \partial c_3$ is a homologous chain then

$$f(c_2) = f(c_1) + f(\partial c_3) = f(c_1) + \delta f(c_3) = f(c_1). \tag{8}$$

Thus the value of $f(c)$ depends only on the homology class of c. This is an analogue of 7.6. Now suppose that c is a cycle and that f_1 and f_2 cohomologous $f_2 = f_1 + \delta f_3$. Then for any cycle c

$$f_2(c) = f_1(c) + \delta f_3(c) = f_1(c) + f_3(\partial c) = f_1(c). \tag{9}$$

Now each cocycle $f \in Z^n(S;\mathcal{R})$ restricts to a linear map $Z_n(S;\mathcal{R}) \to \mathcal{R}$ that is trivial on $B_n(S;\mathcal{R})$ by 7.8. Thus f is defines a linear form on $H_n(S;\mathcal{R}) = Z_n(S;\mathcal{R})/B_n(S;\mathcal{R})$, Hence a linear transformation $Z^n(S;\mathcal{R}) \to \mathrm{Hom}(H_n(S;\mathcal{R}),\mathcal{R})$. Next if f_1 and f_2 are the cohomologous cycles then by 7.9 they have the same image in $\mathrm{Hom}(H_n(S;\mathcal{R}),\mathcal{R})$. If fact, the map is an isomorphism though we do not prove it here.

Proposition 7.15 *Let S be a connected, closed, orientable surface of genus σ, and $\mathcal{R}$. Then, the linear map $H^n(S;\mathcal{R}) \to \mathrm{Hom}(H_n(S;\mathcal{R}),\mathcal{R})$, $f \to f(c)$ is a linear isomorphism.*

Remark 7.10 Note that each differential form ω defines an element of $C^n(T;\mathcal{R})$ by integration $c \to \int_c \omega$. This allows us to identify cohomology defined in the two different ways. It also shows that cohomology is independent of the triangulations.

Functorial properties of cohomology Let $p : S \to T$ be a branched, differentiable covering of connected Riemann surfaces. This means that, except at a finite number of points the differential dp is an isomorphism of tangent spaces, and at the points where dp is not, then in local complex coordinates p is equivalent to $z \to z^t$, or $z \to \overline{z}^t$. The integer t is called the branching order. Our canonical examples will be the quotient map $S \to S/G$ of a conformal group action, and a rational function on a surface S, so that $T = \widehat{\mathbb{C}}$, the Riemann sphere. Given a differential form ω on T it may be pulled back to a differential form $p^*\omega$ on S via p. If p is given by $p(x,y) = (u,v) = (u(x,y), v(x,y))$ in local coordinates x, y on S and u, v on T. Then if $\omega = f(u,v)$ is a function then $p^*\omega$ is the composed function

ω	$p^*\omega$
$\omega = f(u,v)$	$f(u(x,y), v(x,y))$
$\omega = g(u,v)du + h(u,v)dv$	$g(u(x,y), v(x,y))du(x,y) + h(u(x,y), v(x,y))du(x,y)$
	$= g(u,v)(u_x dx + u_y dy) + h(u,v)(v_x dx + v_y dy)$
$\omega = k(u,v)du \wedge dv$	$k(u(x,y), v(x,y))(u_x v_y - u_y v_x)du \wedge dv.$

,

where we have shortened $g(u(x,y),v(x,y))$ to $g(u,v)$, etc., for notational convenience. For the geometric interpretations of cohomology, pick a tiling $\mathcal{T}$ on T so that the critical values of p are vertices. Then for every tile Δ or edge e the pullbacks $p^{-1}(\Delta^\circ)$ and $p^{-1}(e^\circ)$ of the interiors are disjoint unions of tile interiors or edge interiors which are mapped homeomorphically onto Δ° and e° respectively. Now we create the pullback tiling $p^{-1}(\mathcal{T})$ on S by declaring the vertices to be inverse images of vertices and the tiles and edges to be the closures of the components of $p^{-1}(\Delta^\circ)$ and $p^{-1}(e^\circ)$, respectively. Thus for every tile, edge or vertex Δ, e or P in $p^{-1}(\mathcal{T})$ the restricted map p is a homeomorphism from Δ, e or P to $p_*\Delta = p(\Delta)$, $p_*e = p(e)$ or $p_*P = p(P)$, respectively. Thus for each n-chain $a_1c_1 + \cdots + a_kc_k$ on S the chain $p_*(a_1c_1 + \cdots + a_kc_k) = a_1p_*c_1 + \cdots + a_kp_*c_k$ is well defined, and we get a linear map $p_* : C_n(S;\mathcal{R}) \to C_n(T;\mathcal{R})$. The adjoint of this map $p^* : C^n(T;\mathcal{R}) \to C^n(S;\mathcal{R})$ is defined by $p^*(f)(c) = f(p_*(c))$ for each cochain f and chain c. It is easy to show that p_* commutes with the boundary operator $p_*\partial = \partial p_*$, and consequently that $p^*\delta = \delta p^*$. Thus p_* and p^* induce linear maps:

$$p_* : H_n(S;\mathcal{R}) \to H_n(T;\mathcal{R}),\ p^* : H^n(T;\mathcal{R}) \to H^n(S;\mathcal{R}).$$

Remark 7.11 In Remark 3 it was noted that the cohomology via triangulations and differential forms could be related by integration. This interpretation, along with the change of variables formula shows that $p^*\omega$ is the same whether it is computed by differential forms or by a tiling. It is also independent of the tiling chosen on T.

We will freely use in the next chapter the following properties of p_* and p^*.

Proposition 7.16 *Let $p : S \to T$, and $q : T \to U$ be a branched, differentiable covering of connected Riemann surfaces. Then,*

1. $(qp)_* = q_*p_*$, i.e., $p \to p_*$ is covariant,
2. $(qp)^* = p^*q^*$, i.e., $p \to p^*$ is contravariant,
3. $p_* : H_n(S;\mathcal{R}) \to H_n(T;\mathcal{R})$ is surjective,
4. $p^* : H^n(T;\mathcal{R}) \to H^n(S;\mathcal{R})$ is injective.

Remark 7.12 Suppose that G acts conformally on a surface S. Then the quotient $T = S/G$ may be given a tiling $\mathcal{T}$ such that the pull back $p^{-1}(\mathcal{T})$ is a tiling invariant under action of G, just as the kaleidoscopic tilings were invariant under G^*. Then, in a very transparent way, for each $g \in G$ the map $x \to gx$, denoted as $g : S \to S$, induces an automorphism $g_* : H_n(S;\mathcal{R}) \to H_n(S;\mathcal{R})$ in homology, and an automorphism $g^* : H^n(S;\mathcal{R}) \to H^n(S;\mathcal{R})$ in cohomology. The maps $g \to g_*$ and $g \to (g^{-1})^*$ define representations of G on the homology (as we had before) and cohomology respectively. We may speak of the G-invariant homology and cohomology:

$$\begin{aligned} H_n(S;\mathcal{R})^G &= \{c \in H_n(S;\mathcal{R}) : g_*c = c\}, \forall g \in G\}, \\ H^n(S;\mathcal{R})^G &= \{f \in H_n(S;\mathcal{R}) : g^*\omega = \omega\}, \forall g \in G\}. \end{aligned}$$

The homology and cohomology representations are also defined for G^* but the quotient S/G^* is not a Riemann surface and some of the nice properties of the next proposition are lost.

Our last propositions is our main interest in cohomology and the cohomology representation.

Proposition 7.17 *Let the group G act conformally on the surface S, and let $p : S \to T = S/G$, be the quotient map. Then $p^* : H^n(T;\mathcal{R}) \to H^n(S;\mathcal{R})$ maps $H^n(T;\mathcal{R})$ isomorphically onto $H^n(S;\mathcal{R})^G$.*

Remark 7.13 The construction of the homology and cohomology actions and propositions 7.16 and 7.17 do not require G-equivariant tilings to be valid. However, the G-equivariant tilings are useful in constructions and proofs.

7.7 REU Problems

The homology covers and the various modules defined above lead to some interesting but somewhat more advanced problems as more background in homology and group representations is needed.

Problem R7.1 Analyze the oval submodules. Do they contain any interesting information?

Problem R7.2 Do the same for the mirror submodules.

Problem R7.3 When does 7.2 split?

Problem R7.4 How do ovals and mirrors lift under homology covers? In particular are splitting properties preserved.

Problem R7.5 Do binary covers associated to the oval and mirror submodules have anything to do with separability?